Physics 520 – Graduate Quantum Mechanics I
Homework III, due 2:00 p.m. September 12, 2007

This homework has two pages

1. Read G&Y Chap. 2.2(c).

2. Quantum Mechanics Conceptual Survey

   Take the QMCS online at http://www.colorado.edu/sei/surveys/QMCS/520/QMCS-520.html You must come by office hours to get your username and password. This part of the homework is mandatory: you must take this "survey" of your undergraduate quantum understanding to continue in the course. All results are retained anonomously, although you need to enter your student ID for me to verify that you've done this. The survey should take no more than one hour.

3. Spin-1/2 Basis Transformation

   Construct $|S_{n\pm}\rangle$ in the basis $\beta = \{|S_{z+}\rangle, |S_{z-}\rangle\}$ such that $\hat{S}_{n\pm}|S_{n\pm}\rangle = \pm\hbar/2|S_{n\pm}\rangle$. Give a clear geometrical interpretation for your answer. Then construct the operator $\hat{S}_n$, again in basis β. You may use any either matrix or ket representation.

4. Physical Application of Two-state Systems: The Ammonia Molecule

   First read Le Bellac Sec. 5.3 up through Eq. (5.50). In case you cannot borrow Le Bellac from a colleague, I have provided an excerpt on my door which you can photocopy. The Hamiltonian $\hat{H}$ is the operator which corresponds to the physically measurable quantity of the total energy in conservative systems: $\hat{H}|E\rangle = E|E\rangle$. After thinking about the experimental context carefully, return to Eq. (5.47), the Hamiltonian which describes the response of the neutral Ammonia molecule to a static electric field, and perform the following calculations.

   (a) Diagonalize this Hamiltonian exactly. Write down the eigenvalues and eigenvectors.

   (b) Determine the eigenvectors and eigenvalues in the limit of large polarizing field $\mathcal{E}$, i.e., $d\mathcal{E} \gg A$, to lowest non-zero order in A. Explain your result physically.

   (c) Determine the eigenvectors and eigenvalues in the limit of small polarizing field $\mathcal{E}$, i.e., $d\mathcal{E} \ll A$, to lowest non-zero order in $\mathcal{E}$. Explain your result physically.

   (d) Are your eigenvectors consistently orthogonal as you expand in the small parameter in (b) and (c)? Why or why not? Show this explicitly to lowest non-zero order.

   (e) With a numerical tool such as *Mathematica*, plot the eigenvalues as a function of $d\mathcal{E}$, taking the zero of energy to be E_0 and the units of $d\mathcal{E}$ to be A. Show on your plot that your limits from (b) and (c) are reproduced explicitly. Do your eigenvalues ever cross? What would it mean physically if they did?

5. Compatability and Degeneracy

   Consider a three-dimensional ket space. If a certain set of orthonormal kets – say, $|1\rangle$, $|2\rangle$, and $|3\rangle$ – are used as base kets, the operators $\hat{A}$ and $\hat{B}$ are represented by

$$[\hat{A}]_\beta = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad [\hat{B}]_\beta = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & -a \end{pmatrix}, \tag{1}$$

   with both a and b real, in the same basis β.

(a) The operator $\hat{A}$ exhibits a degenerate spectrum. Does $\hat{B}$ also exhibit a degenerate spectrum?

(b) Show that $\hat{A}$ and $\hat{B}$ are compatible. Is your result representation independent, i.e., independent of your choice of basis β? Why or why not?

(c) Find a new set of orthornomal kets which are simultaneous eigenkets of $\hat{A}$ and $\hat{B}$. Specify the eigenvalues of $\hat{A}$ and $\hat{B}$ for each of the three eigenkets. Does your specification of eigenvalues completely characterize each eigenket?

6. The Determinant and the Trace

(a) Let a matrix $A(t)$ depending on parameter t satisfy

$$\frac{d}{dt}A(t) = A(t)B\,, \tag{2}$$

where B is also a matrix. Prove that $A(t) = A(0)\exp(Bt)$. What is the solution of

$$\frac{d}{dt}A(t) = BA(t)\,? \tag{3}$$

(b) Show that

$$\det e^{At_1} \det e^{At_2} = \det e^{A(t_1+t_2)}\,. \tag{4}$$

Then derive the relation

$$\det e^{A} = e^{\mathrm{Tr}\,A} \tag{5}$$

or, equivalently,

$$\det B = e^{\mathrm{Tr}\ln B}\,. \tag{6}$$

You can assume that all A and B are Hermitian matrices for this part.

(c) (Extra credit only) Prove result (b) for *non*-Hermitian matrices.