Physics 520 – Graduate Quantum Mechanics I
Homework IX, due 2:00 p.m. November 7, 2007

This homework has two pages

1. **Reading**

   G&Y Chap. 3.5, all subsections.

2. **Vector Operators**

   Consider a vector operator $\hat{\vec{V}}$.

   (a) Prove the property

$$[\hat{\vec{J}}^2, \hat{\vec{J}} \times \hat{\vec{V}}] = 2i\hbar \left\{ \hat{\vec{J}}^2 \hat{\vec{V}} - (\hat{\vec{J}} \cdot \hat{\vec{V}})\hat{\vec{J}} \right\} \tag{1}$$

   (b) Demonstrate that

$$\begin{aligned} \langle jm'|\hat{\vec{V}}|jm\rangle &= \frac{1}{\hbar^2 j(j+1)} \langle jm'|(\hat{\vec{J}} \cdot \hat{\vec{V}})\hat{\vec{J}}|jm\rangle \\ &= \frac{1}{\hbar^2 j(j+1)} \langle jm'|\hat{\vec{J}}|jm\rangle\langle jm|(\hat{\vec{J}} \cdot \hat{\vec{V}})|jm\rangle \end{aligned} \tag{2}$$

   (c) Assume now that the states $|jm\rangle$ correspond to two angular momentum operators $\hat{\vec{J}}_1$, $\hat{\vec{J}}_2$, i.e., $|jm\rangle \to |j_1 j_2 jm\rangle$. Calculate the diagonal matrix elements $\langle j_1 j_2 jm|\hat{\vec{J}}_1|j_1 j_2 jm\rangle$ and $\langle j_1 j_2 jm|\hat{\vec{J}}_2|j_1 j_2 jm\rangle$.

3. **Visualizing Angular Momentum 1 in Position Space**

   Consider a state with angular momentum $j = l = 1$:

$$|\psi\rangle = c_{-1}|1,-1\rangle + c_0|1,0\rangle + c_1|1,1\rangle\,. \tag{3}$$

   (a) Find a direction $\hat{e}_n$ such that this state is an eigenstate of the operator $\hat{L}_n \equiv \hat{\vec{L}} \cdot \hat{e}_n$. Express the coefficients c_i in terms of the angles θ, ϕ that define the direction of $\hat{e}_n$ in spherical coordinates.

   (b) Write down expressions for the eigenvectors of $\hat{L}_y$ and $\hat{L}_x$.

4. **Spin One Time Evolution**

   Consider a particle with spin quantum number $s = 1$. Ignore all spatial degrees of freedom and assume the particle is subject to an external magnetic field $\vec{B} = B\hat{e}_x$. The Hamiltonian is

$$\hat{H} = g\vec{B} \cdot \hat{\vec{S}}\,. \tag{4}$$

   (a) Obtain explicitly the spin matrices in the basis $|s, m_s\rangle$.

   (b) If the particle is initially in the state $|1,1\rangle$, find the evolved state of the particle at times $t > 0$.

   (c) What is the probability of finding the particle in state $|1,-1\rangle$ as a function of time?

5. **Free Particle in Three Dimensions**

A particle has the wave function $\psi(r) = N \exp(-\alpha r)$ where N is a normalization factor and α is a known real parameter.

(a) Calculate the factor N.

(b) Calculate the expectation values $\langle \hat{\vec{x}} \rangle$, $\langle \hat{r} \rangle$, $\langle \hat{r}^2 \rangle$ in this state.

(c) Calculate the uncertainties $\langle (\Delta \hat{\vec{x}})^2 \rangle$ and $\langle (\Delta \hat{r})^2 \rangle$.

(d) Calculate the probability of finding the particle in the region $r > \Delta r \equiv \sqrt{\langle (\Delta \hat{r})^2 \rangle}$.

(e) What is the momentum space wavefunction $\tilde{\psi}(\vec{k}, t)$ at any time $t > 0$?

(f) Calculate the uncertainty $\langle (\Delta \hat{\vec{p}})^2 \rangle$.

(g) Show that the wave function is at all times isotropic, i.e., $\psi(\vec{x}, t) = \psi(r, t)$. How does the expectation value $\langle \hat{\vec{x}} \rangle$ evolve in time?