Physics 520 – Graduate Quantum Mechanics I
Homework IV, due 2:00 p.m. September 19, 2007

This homework has two pages

1. Read G&Y Chap. 2.2(d)-(e) and 2.3(a).

2. Density Operator Basics

   (a) Write down $\hat{S}_{1z} \otimes \hat{S}_{2z}$ in operator form in the $\{|\epsilon_1\rangle \otimes |\epsilon_2\rangle\}$ basis, where $\epsilon_1 \in \{\pm 1\}$, $\epsilon_2 \in \{\pm 1\}$ refer to the eigenstates of $\hat{S}_{1z}$ and $\hat{S}_{2z}$, respectively.

   (b) Calculate the density operator $\hat{\rho}$ corresponding to pure state $|\psi\rangle = a|-\rangle + b|+\rangle$ for a single qubit.

   (c) Show that the determinant of this operator vanishes.

   (d) Show that $\hat{\rho} = \hat{\rho}^2$.

   (e) In what basis is this operator diagonal? Perform the diagonalization explicitly.

3. Spin-1/2 Density Operator

   (a) Consider a pure system of identically prepared spin-1/2 systems. Suppose the expectation values $\langle \hat{S}_x \rangle$, $\langle \hat{S}_z \rangle$, and the sign of $\langle \hat{S}_y \rangle$ are known. Show how we may determine the state vector. Why is it unnecessary to know the magnitude of $\langle \hat{S}_y \rangle$?

   (b) Now consider a *mixed* ensemble of spin-1/2 systems. Suppose the expectation values $\langle \hat{S}_x \rangle$, $\langle \hat{S}_y \rangle$, and $\langle \hat{S}_z \rangle$ are all known. Construct the 2×2 density operator that characterizes the ensemble.

4. Spin-1/2 Thermal Ensembles

   Consider a thermal ensemble of spin-1/2 electrons. The Hermitian operator corresponding to the physically measurable projection of the magnetic moment of the electron in the $\hat{e}_z$ direction is

$$\hat{\mu}_z = -\frac{e}{m_e c} \hat{S}_z \, . \tag{1}$$

   The Hamiltonian corresponding to the physically measurable energy is

$$\hat{H} = \frac{\hbar\omega}{2} \left(|S_{z+}\rangle\langle S_{z+}| - |S_{z-}\rangle\langle S_{z-}| \right) , \tag{2}$$

   where $\omega \equiv |e|B/m_e c$ and B is a uniform magnetic field in the $\hat{e}_z$ direction.

   (a) For a single electron in the ensemble, determine the density operator $\hat{\rho}$ corresponding to a thermal probability distribution characterized by temperature T.

   (b) Calculate the expectation value of $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ for a single electron in the ensemble.

   (c) Calculate the average z-magnetic moment μ_z per electron, and plot your result. Make sure to indicate on your plot where the electron is completely aligned with the magnetic field, and where the orientation is completely random.

5. Schrödinger Kitten

   Consider a density operator formed from the pure GHZ state for three qubits, $\hat{\rho} = |\psi\rangle\langle\psi|$, where $|\psi\rangle \equiv \frac{1}{\sqrt{2}}(|+++\rangle - |---\rangle)$.

(a) Explicitly calculate *by hand* the reduced density operator $\hat{\rho}_{12} \equiv \mathrm{Tr}_3\hat{\rho}$. Interpret your result physically.

(b) Calculate by hand $\hat{\rho}_1 \equiv \mathrm{Tr}_2\hat{\rho}_{12}$. Interpret your result physically.

(c) What are $\hat{\rho}_2$, $\hat{\rho}_3$, $\hat{\rho}_{23}$, and $\hat{\rho}_{13}$?

(d) Calculate the *generalized entanglement* (quantum Tsallis entropy) for the GHZ state, defined by

$$Q \equiv 2\left(1 - \frac{1}{N}\sum_{n=1}^{N} \mathrm{Tr}\hat{\rho}_n^2\right) . \tag{3}$$

Here $N = 3$, since there are 3 qubits, and $\hat{\rho}_N$ is the reduced density operator you calculated in (b) and (c).

(e) The quantity $\mathrm{Tr}\hat{\rho}_n^2$ is called the *purity*. Why was the name "purity" chosen for this quantity? Determine the upper and lower limits of Q. What are the corresponding limits on the purity? To what do these limits correspond to physically?

(f) Calculate the *entropy of entanglement* for each q-bit given by

$$\sigma_n \equiv -\mathrm{Tr}\hat{\rho}_n \log_2 \hat{\rho}_n \, . \tag{4}$$

Again, $\hat{\rho}_n$ is the reduced state operator. The entropy of entanglement is another common entanglement measure.