Physics 520 – Graduate Quantum Mechanics I
Homework VI, due 2:00 p.m. October 10, 2007

This homework has two pages

1. **Reading**

   G&Y Chap. 2.5(a)-(c).

2. **Free Particle in One Spatial Dimension**

   (a) Show that a Gaussian minimizes the uncertainty product $\langle(\Delta\hat{x})^2\rangle\langle(\Delta\hat{p})^2\rangle$. Find the form of the Gaussian and describe all parameters physically.

   (b) Show that your Gaussian is a solution of the free particle Hamiltonian in one spatial dimension, $\hat{H} = \hat{p}^2/2m$. Writing your wavepacket in the form

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} f(k) \exp[i(kx - E_k t/\hbar)] \tag{1}$$

   where $E_k = \hbar^2 k^2/2m$, find the form of the weighting function $f(k)$.

   (c) Find the probability amplitude and probability as a function of time.

   (d) Find the wavepacket Gaussian width Δx as a function of time. How does the width evolve forwards and backwards in time?

3. **Snell's Law in the Classically Allowed Regime**

   Consider a particle incident on an infinite planar surface separating empty space and an infinite region with constant potential energy V. The energy of the particle is $E > V$. Choose coordinates such that

$$V(x) = \begin{cases} 0 & x < 0 \\ V & x \geq 0 \end{cases} . \tag{2}$$

   The incident particle is represented by a wave function of plane-wave form,

$$\psi_i(\mathbf{x},t) = A\exp(i\mathbf{k}\cdot\mathbf{x} - \omega t) . \tag{3}$$

   The reflected and transmitted wave functions are

$$\begin{aligned} \psi_r(\mathbf{x},t) &= B\exp(i\mathbf{k}'\cdot\mathbf{x} - \omega t) , & (4)\\ \psi_t(\mathbf{x},t) &= C\exp(i\mathbf{q}\cdot\mathbf{x} - \omega t) . & (5)\\ & & (6) \end{aligned}$$

   The incident, reflected, and transmitted *wave vectors* are

$$\begin{aligned} \mathbf{k} &= k(\hat{e}_x\cos\theta + \hat{e}_y\sin\theta) , & (7)\\ \mathbf{k}' &= k'(-\hat{e}_x\cos\theta' + \hat{e}_y\sin\theta') , & (8)\\ \mathbf{q} &= q(\hat{e}_x\cos\theta'' + \hat{e}_y\sin\theta'') . & (9)\\ & & (10) \end{aligned}$$

   (a) Show that the angle of reflection equals the angle of incidence, i.e., $\theta' = \theta$. Show the validity of Snell's law,

$$\frac{\sin\theta''}{\sin\theta} = n , \tag{11}$$

where n is the index of refraction of the $x > 0$ region.

(b) Compute the coefficient ratios B/A and C/A.

(c) Calculate the incident, reflected, and transmitted probability current densities $\mathbf{j}_i$, $\mathbf{j}_r$, and $\mathbf{j}_t$.

(d) Demonstrate that the component of current perpendicular to the $x = 0$ plane is conserved.

(e) Compute the transmission and reflection coefficients, defined as

$$T \equiv \frac{\mathbf{j}_t \cdot \hat{e}_x}{\mathbf{j}_i \cdot \hat{e}_x}, \quad (12)$$

$$R \equiv \frac{\mathbf{j}_r \cdot \hat{e}_x}{\mathbf{j}_i \cdot \hat{e}_x}. \quad (13)$$

Show that $R + T = 1$.

4. **Snell's Law in the Classically Forbidden Regime**

Consider the same problem as before, but for $E < V$. Then the incident and reflected wave function and wave vectors are the same. However, the transmitted wave function takes the form

$$\psi_t(\mathbf{x}, t) = C \exp(\kappa x - iqy - i\omega t). \quad (14)$$

(a) Show that $k' = k$, $\theta = \theta'$, and $q = k \sin\theta$, the analog of Snell's law.

(b) Compute the coefficient ratios B/A and C/A.

(c) Calculate the incident, reflected, and transmitted probability current densities.

(d) Demonstrate that the component of current perpendicular to the $x = 0$ plane is conserved.

(e) Explicitly compute the reflection coefficient and verify that it equals unity.

Remark: the same effect occurs for evanescent waves with light.