Physics 521 – Graduate Quantum Mechanics
Homework III, due Wednesday, January 30, 2008 at 8:30 a.m.

This homework has two pages.

1. **Reading**

   G&Y Section 4.3

2. **Propagator for the harmonic oscillator**

   Consider the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \,, \tag{1}$$

   i.e., the simple harmonic oscillator in 1D. Take as the initial state

$$\langle x|\alpha, t_0 = 0\rangle = \pi^{-1/4} \ell^{-1/2} \exp(-x^2/2\ell^2) \,, \tag{2}$$

   where $\ell = 2x_0 \equiv 2\sqrt{\hbar/m\omega}$ is the width.

   (a) How does the initial state differ from the usual harmonic oscillator ground state?

   (b) Using the Feynman propagator, evaluate the probability density

$$\mathcal{P}(x,t) \equiv |\langle x|\alpha, 0; t\rangle|^2 \tag{3}$$

   at one-eighth intervals of the period, i.e., for $t = \frac{T}{8}, \frac{T}{4}, \frac{3T}{8}, \frac{T}{2}, \frac{5T}{8}, \frac{3T}{4}, \frac{7T}{8}, T$, where $T \equiv \frac{2\pi}{\omega}$.

   (c) Plot the time dependence of the second moment of the distribution,

$$\langle \hat{x}^2 \rangle(t) \equiv \int_{-\infty}^{+\infty} dx\, x^2\, \mathcal{P}(x,t) \,. \tag{4}$$

   Both numerical and analytical results are acceptable. Physically motivate and interpret your result.

3. **Harmonic oscillator basics**

   A simple harmonic oscillator is initially in a state with position representation

$$\psi_\alpha(x, t=0) = N \sum_{n=0}^{\infty} c^n \psi_n(x) \tag{5}$$

   where the $\psi_n(x)$ are the harmonic oscillator energy eigenfunctions and c is a complex scalar.

   (a) Calculate the normalization constant N.

   (b) Find the state of the system in position representation at time $t > 0$.

   (c) Calculate the probability of finding the system again in the initial state at a later time $t > 0$.

   (d) Compute the expectation value of the energy.

4. **Oscillation of a charged particle in a uniform electric field**

   A particle of mass m and electric charge q can move only in one dimension and is subject to a harmonic force a homogeneous electric field. The Hamiltonian operator for this system is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 - q\mathcal{E}\hat{x} \,. \tag{6}$$

Please do not use perturbation theory to solve this problem, as it is unnecessary.

(a) Solve the energy eigenvalue problem.

(b) If the system is initially in the ground state of the unperturbed harmonic oscillator ground state, $|\psi(0)\rangle = |0\rangle$, i.e., the ground state for no electric field, what is the probability of finding it in the ground state of the *full* Hamiltonian.

(c) Assume again that the system is initially in the unperturbed harmonic oscillator ground state and calculate the probability of finding it in this state again at a later time.

(d) With the same initial condition calculate the probability of finding the particle at a later time in the first excited state of the unperturbed harmonic oscillator. I.e., what is the time-dependence of the probability that the electric field causes a transition from the ground state to the first excited state.

(e) Consider the electric dipole moment $d \equiv qx$ and calculate its vacuum expectation value in the evolved state $|\psi(t)\rangle$, assuming again that we start from the unperturbed state at $t = 0$.