Physics 521 – Graduate Quantum Mechanics
Homework VIII, due Friday, March 21, 2008 at 9:00 a.m.

This homework has two pages.

1. **Reading**

   G&Y Sections 7.1 through 7.4

2. **Electron in a Square Well**

   An electron is in the $n = 1$ eigenstate of a one-dimensional infinite square-well potential which extends from $x = -a/2$ to $x = a/2$. At $t = 0$ a uniform electric field E is applied in the x direction. It is left on for a time τ and then removed. Use time-dependent perturbation theory to calculate the probabilities P_2 and P_3 that the electron will be, respectively, in the $n = 2$ and $n = 3$ eigenstates at $t > \tau$. Assume that τ is short in the sense that $\tau \ll \frac{\hbar}{E_1 - E_2}$, where E_n is the energy of the eigenstate n. Specify any requirements on the parameters of the problem necessary for the validity of the approximations made in the application of time-dependent perturbation theory.

3. **Propagator in Imaginary Time**

   A particle of mass m moves in one dimension under the influence of forces given by the double-well potential

$$V(x) = \frac{m\omega^2}{4a^2}(x^2 - a^2)^2 \,. \tag{1}$$

(a) Consider the propagator and replace the time variable t by the imaginary time $\tau \equiv it$ (sometimes called the Euclidean time). Then show in general that the lowest energy eigenvalue can be obtained from the limit

$$E_0 = -\lim_{\tau\to\infty} \frac{\hbar}{\tau} \ln[K(x', x; -i\tau)] \,. \tag{2}$$

(b) For the potential given above, obtain a solution of the classical equations of motion in imaginary time. Let the particle start at the local minimum $-a$ at $\tau = -T/2 \to -\infty$ and ends up at the other local minimum, $x' = a$, at $\tau = T/2 \to \infty$. Discuss also the possibility of a solution that is the sum of two such solutions with widely separated centers.

(c) Consider a classical solution that is the sum of N widely separated solutions of the type derived in (b). In the limit $\hbar \to 0$ each of these solutions gives an approximation to the propagator through the formula

$$K(a, -a; -iT) \simeq \sum_N A_N \exp(-S_c^{(N)}[x]/hbar) \,, \tag{3}$$

where A_N is an unknown constant and $S_c^{(N)}[x]$ is the classical action in imaginary time for each of the solutions. Derive the dependence on N of each of the above quantities and perform the summation.

(d) Calculate the ground state energy of the system in terms of the given parameters and the unknown factor A_N.

Hint: Recall the classical action is $S_c = \int_{-T/2}^{T/2} d\tau \left[\frac{1}{2}m\dot{x}^2 + V(x)\right]$.

4. **Lyman Alpha Line**

Consider a hydrogen atom in a $2p$ state that is perturbed by a plane electromagnetic wave of wave number $\vec{k}$ and frequency $\omega = ck$:

$$\vec{A}(\vec{r}, t) = 2\vec{A}_0 \cos(\omega t - \vec{k} \cdot \vec{r}) \,. \tag{4}$$

The positive frequency part of this vector potential gives a semi-classical description of the emission of a photon. Assume that the wavelength $\lambda = 2\pi/k$ is much larger than the effective dimension of the atom a_0.

(a) Calculate the probability per unit time of finding the atom in its ground state ($n = 1, \ell = m = 0$) in first-order perturbation theory, i.e., for photon emission via $2p \to 1s + \gamma$. Assume that we are interested in times $t \gg \omega^{-1}$ and $t \gg \hbar/|E_1|$. Express your result in terms of the matrix element of the electric dipole moment of the atom.

(b) Integrate over all possible wave numbers $\vec{k}$ of the electromagnetic wave and obtain the $2pto1s$ transition rate per unit solid angle corresponding to the direction of the emitted photon $\hat{e}_k$. Determine the amplitude A_0 from the condition that the energy density calculated from the above electromagnetic wave must coincide with the energy density corresponding to one photon per unit volume.

(c) Obtain the total transition rate assuming that we do not measure the direction and polarization of the emitted photon.