1 Overview: Network Equilibria

The article “Equilibrium statistical mechanics of network structures” (I. Farkas, I. Derényi, G. Palla, and T. Vicsek; 2004) intends to develop the fundamentals of a statistical physics on abstract networks. It first determines what, exactly, “equilibrium” means for a network, then gives a broad review of networks in equilibrium. Their review includes possible definitions for energies and ensembles, methods to construct and modify certain classes of networks, analyses of topology and phase transitions, and brief examples of physical applications. While the authors achieve their stated intent, they do not clearly address methods for more practical networks – e.g., graphs with weighted edges or “loop” edges. As such, it seems that more work needs to be done.

The article’s premise: a network has a fixed number of vertices, and edges act as the “particles” of the system; and possible graphs in the network are assumed to be unweighted, simple and undirected. Two fundamental problems of modeling a physical concept within an abstract construct are addressed:

- What is “equilibrium”?
- What is “energy”?

The first is resolved by means of a definition: “equilibrium network ensembles [are] stationary ensembles of graphs generated by restructuring processes obeying detailed balance and ergodicity.” More clearly:

Def Assume a network changes over time. Denote the transition rate between graph a and graph b as $r_{a \to b}$. The time evolution of the probability distribution for each graph, then, can be written as

$$\frac{\partial P_a}{\partial t} = \sum_b (P_b r_{b \to a} - P_a r_{a \to b}),$$

where P_a is the probability of graph a. Then we say **a network is at equilibrium** whenever the following hold:

- $r_{a \to b}$ is nonzero for any two graphs a and b
- The system converges over time to a stationary distribution P_a^{stat} for each graph a
- For each graph a, P_a^{stat} satisfies $P_a^{stat} r_{a \to b} = P_b^{stat} r_{b \to a}$.

The second problem, that of defining “energy” on a network, has no correct answer in an abstract system. Instead the authors provide examples of pre-established network energies, and their role in statistical ensembles.

2 Network Energies and Ensembles

Several instructional examples are given of network energy and statistical network ensembles. These are best summarized in tables. *Note:* k_i is the degree of vertex i; s_i is the size of the ith component of the graph; and $d_{i,j}$ is the shortest distance from vertex i to vertex j.

Type	Generic: $E = \ldots$	Example	Notes
Vertex-degree	$\sum_{i=1}^{N} f(k_i)$	$\sum_{i=1}^{N} \left[-\frac{k_i^2}{2} + \mu k_i^3 \right]$	As $T \to \infty$, any f produces the random graph ensemble. The example prevents condensation of edges at low T.
Edge-degree	$\sum_{(i,j)} f(k_i, k_j)$	$\sum_{(i,j)} \frac{\min(k_i, k_j)}{\max(k_i, k_j)} - 1$	Can tune favorability of degree-degree correlations over arbitrary distances
Component Size	$\sum_{i=1}^{n} f(s_i)$	$-\max_i s_i$	*None*
Connectivity	$\sum_{i,j} f(d_{i,j})$	$\sum_{i,j} d_{i,j}$	Useful for optimizing graph "diameter"
a priori $\{P_a\}$	Energy is determined by a known probability distribution. See Ensembles table.		

Definitions of the classical ensembles are straightforward, for a number of edges M, a chosen temperature T, a total energy E, and arbitrary constants ϵ, μ:

Ensemble	Meaning	Normalized P_a	Partition function Z	Energy for *a priori* $\{P_a\}$
Microcanonical	M, E fixed	Z^{-1}	$\Omega(M, E)$	ϵ
Canonical	M fixed, E varies	$\frac{e^{-E_a/T}}{Z}$	$\sum_b e^{-E_b/T}$	$-T \log P_a + \log Z$
Grand Canonical	M, E vary	$\frac{e^{-(E_a - \mu M_a)/T}}{Z}$	$\sum_b e^{-(E_b - \mu M_b)/T}$	$-T \log P_a + \mu M_a + \log Z$

Little is said about the ensembles, as the intention seems to be merely to establish a mapping from network ensembles to classical statistical mechanical ensembles. The energies are, of course, incomplete but those listed are accompanied by helpful tidbits about their effects on network behavior.

3 Constructions & Modification Processes

The article includes an implicit review of several well-known network types and how they fit into the statistical view. This table summarizes:

Network	Ensemble	Construction	Modification	Notes
Random [constant M]	Microcanonical	M edges placed randomly and independently between vertices	Randomly move an edge	$p(k)$ converges to a Poisson distribution as the number of vertices $N \to \infty$.
Random [constant $\langle M \rangle$]	Grand Canonical	Connect each pair of vertices with an edge with probability p	Randomly choose two vertices, connect with probability p (or disconnect with probability $1-p$)	
Small-World Graph	Canonical	Order vertices in a list, then for each vertex connect its k nearest neighbors	Move each edge to a random pair of vertices with fixed probability r	*None*
Constant Degree-Distribution $p(k)$	Canonical	*None given*	**link randomization:** Select 2 edges randomly, swap endpoints. **vertex randomization:** Select 2 vertices randomly, select an edge at each vertex randomly, swap selected vertices as endpoints.	The degree distribution $p(k)$ is the probability that a vertex has degree k. Vertex randomization generates positive degree-degree correlations.
Constant Degree-Degree Correlation $p(k,k')$	Any	For each vertex i, choose degree q_i from the p_k distribution. Then for each vertex pair i,j, place the edge (i,j) with probability $\frac{\langle k \rangle p(q_i,q_j)}{N p_{q_i} p_{q_j}}$	*None given*	$p(k,k')$ is the probability that the endpoints of an edge have degrees k and k'. Fixing $p(k,k')$ determines $p(k)$ through the relation $\sum_{k'} p(k,k') = \frac{k p_k}{\langle k \rangle}$

4 Topology & Phase Transitions

The **topology** of a graph is defined as the set of graphs which are identical to that graph up to a permutation of their vertices. By measuring macroscopic quantities of topologies such as $\langle k \rangle$ the average vertex degree, $p(k)$ the degree distribution, $p(k, k')$ the degree-degree correlation distribution, s_{max} the largest component size, etc., the authors find that networks tend strongly toward certain topologies (*phases*) in precise temperature ranges, and that *topological phase transitions* occur when the temperature crosses the boundaries of these ranges.

The authors included diagrams of the phase transitions for a handful of different energies and ensembles. The table below gives a good summary of these, but the diagrams in the paper are better:

$E = \ldots$	Macroscopic Observables	Phase Transitions
$-\sum_i \frac{k_i^2}{2}$	$\phi = k_{max}/M$	$T \to \infty$ always converges to the random graph, and E scales as N. At low T, edges condense onto one vertex and E scales non-extensively as N^2.
$-\sum_i k_i \log(k_i)$	k_{max}	$k_{max} \in \begin{cases} \mathcal{O}(\sqrt{M}) \text{for } 0 \leq T \lesssim 0.5 \\ \mathcal{O}(M) \quad \text{for } 0.5 \lesssim T \leq 0.85 \\ \mathcal{O}(1) \quad \text{for } 0.85 \leq T \end{cases}$
$\sum_{(i,j)} \left[\frac{\min(k_i, k_j)}{\max(k_i, k_j)} - 1 \right]$	Unspecified	Classical Random Graph for $T \to \infty$ 'Scale-Free Graph' for T 'intermediate' Condensated for $T \to 0$
$-s_{max}$	$\phi = s_{max}/M$	$\phi = \begin{cases} 1 \text{ for } 1/T > [\langle k \rangle - 1 - \log(\langle k \rangle)] \\ 0 \text{ for } 1/T < [\langle k \rangle - 1 - \log(\langle k \rangle)] \end{cases}$

The authors note that the described phase transitions are *continuous* except in the case of the transition from the $\mathcal{O}(\sqrt{M})$ phase to the $\mathcal{O}(M)$ phase in the $E = -\sum_i k_i \log(k_i)$ case.

5 Applications & Degenerate Graphs

The article makes a weak attempt at explaining how to apply these methodologies for simple unweighted graphs to degenerate weighted graphs. In this aspect the authors are largely unsuccessful and definitely unhelpful, although some independent thought would surely lead to satisfactory solutions.

Similarly, weak attempts are made to demonstrate the use of equilibrium networks in modeling physical systems (polymers, biological networks, etc.). However, one example provides a complete and satisfactory explanation, where a network is used to model a lattice gas. This is done by mapping a graph with N vertices onto a lattice gas with $N(N-1)/2$ lattice sites (i.e., each possible edge of the graph corresponds to a site in the lattice), and applying an appropriate energy. It is shown that using the $-\sum_i k_i^2$ energy is equivalent to "an Ising model with Kawasaki-type dynamics" (not sure what that means).

A more apparent application (which the authors mention only as a side-note) is the choice of E so that a desired property will be optimized quickly using a Monte-Carlo process.