Application of Wavelet Transforms for Solving Integral Equations that arise in Rough Surface Scattering

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Abstract

We consider the problem of scattering a plane wave from a periodic rough surface. The scattered field is evaluated once the field on the boundary is calculated. The latter is the solution of an integral equation. In fact, different integral equation formulations are available in both coordinate and spectral space. We solve these equations using standard numerical techniques and compare the results to corresponding solutions of the equations using wavelet transform methods for sparsification of the impedance matrix. Using an energy check, the methods are shown to be highly accurate. We limit the discussion in this paper to the Dirichlet problem (scalar) or TE-polarized case for a one-dimensional surface. The boundary unknown is thus the normal derivative of the total (scalar) field or equivalently the surface current. We illustrate two conclusions. First, sparsification (using thresholded wavelet transforms) can significantly reduce accuracy. Second, the wavelet transform did not speed up the overall solution. For our examples the solution time was considerably increased when thresholded wavelet transforms were used.

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1 Introduction

We treat the problem of a scalar field $\psi^{\mathrm{i}}(\mathbf{x})$, $(\mathbf{x} = (x, z))$, incident on a periodic one-dimensional rough surface $z = s(x)$ with period L, $s(x + L) = s(x)$. See Fig. 1. The field scattered from the surface $\psi^{\mathrm{sc}}(\mathbf{x})$ satisfies the scalar Helmholtz equation above the surface

$$(\partial_i \partial_i + k_1^2)\psi^{\mathrm{sc}}(\mathbf{x}) = 0, \quad z > s(x). \tag{1}$$

For the examples considered in this paper the total field on the surface $(\mathbf{x}_s = (x, s(x)))$,

$$\psi^{\mathrm{T}}(\mathbf{x}_s) = \psi^{\mathrm{i}}(\mathbf{x}_s) + \psi^{\mathrm{sc}}(\mathbf{x}_s) = 0, \tag{2}$$

which is a Dirichlet (acoustic soft) or TE-polarized electromagnetic boundary condition. Further, we consider only plane wave incidence,

$$\psi^{\mathrm{i}}(\mathbf{x}) = e^{ik_1(\alpha_0 x - \beta_0 z)}, \tag{3}$$

where $k_1 = \frac{2\pi}{\lambda}$ is the wavenumber, λ the wavelength, and $\alpha_0 = \sin(\theta^{\mathrm{i}}), \beta_0 = \cos(\theta^{\mathrm{i}})$, with θ^{i} the angle of incidence. Since the surface is periodic the Bragg condition is in effect

$$\alpha_n = \alpha_0 + n\frac{\lambda}{L}, \tag{4}$$

where n is an integer and $\alpha_n = \sin(\theta_n)$ with θ_n the angle of the n^{th} Bragg mode. The scattered field is thus a set of discrete modes

$$\psi^{\mathrm{sc}}(\mathbf{x}) = \sum_{n=-\infty}^{\infty} A_n e^{ik_1(\alpha_n x + \beta_n z)}, \tag{5}$$

where the unknown amplitudes A_n must satisfy the normalized energy relation

$$\sum_j |A_j|^2 \frac{\mathrm{Re}(\beta_j)}{\beta_0} = 1, \tag{6}$$

which is used as a (necessary but not sufficient) check on the accuracy of our calculations.

2 Integral Equations

There are various formulations of integral equations used to solve this problem [1, 2, 3, 4, 5]. The first is written as

$$F^{\mathrm{i}}(x) = \int_{-\frac{L}{2}}^{\frac{L}{2}} Z_{p_1}^{\mathrm{D}}(x, x') N^{\mathrm{T}}(x') dx', \tag{7}$$

where N^{T} is the normal derivative of the field on the surface (or the surface current), F^{i} a linear combination of the incident field and its normal derivative N^{i} evaluated on the surface (written as a function of only the scalar variable x)

$$F^{\mathrm{i}}(x) = \beta\psi^{\mathrm{i}}(\mathbf{x}_s) + \alpha N^{\mathrm{i}}(\mathbf{x}_s), \tag{8}$$

and the "impedance" kernel (D=Dirichlet)

$$Z^{\mathrm{D}}_{p_1}(x, x') = \frac{1}{2}\alpha\delta(x - x') + \alpha\, \mathrm{PV}\, G'_{p_1}(x, x') + \beta G_{p_1}(x, x'), \tag{9}$$

given by a Dirac delta function term, the periodic Green's function,

$$G_{p_1}(x, x') = \frac{i}{4\pi}\frac{\lambda}{L}\sum_{j=-\infty}^{\infty}\frac{1}{\beta_j}e^{ik_1[\alpha_j(x-x')+\beta_j|s(x)-s(x')|]}, \tag{10}$$

and it normal derivative

$$G'_{p_1}(x, x') = [n_j\partial_j G_{p_1}(\mathbf{x}, \mathbf{x}')]_{\substack{|z=s(x)\\ |z'=s(x')}}. \tag{11}$$

Here n_j is the (non-unit) surface normal and PV symbolically represents Cauchy Principal Value (if necessary). For a detailed derivation we refer to [2, 3].

Discretization of Eq. (7) yields a matrix whose row and column entries both result from sampling in coordinate space, and we refer to the method as coordinate-coordinate (CC). The real parameters α and β specify the type of equation. For $\alpha = 0$ we have a first kind integral equation referred to as CC1, for $\beta = 0$ the result is a second kind integral equation CC2, and for $\beta = 1$ and α arbitrary we obtain the combined field integral equation (CFIE) [6, 7, 8, 9]. Solution of the unknown N^{T} yields the amplitudes A_n. See the discussion below.

The second formulation is given by the set of equations

$$P_0^{\pm}(\alpha_n) = F^{\pm}(\alpha_n), \tag{12}$$

where

$$P_0^{\pm}(\alpha_n) = \frac{1}{L}\int_{-\frac{L}{2}}^{-\frac{L}{2}} e^{-ik_1[\alpha_n x \mp \beta_n s(x)]}N(x)dx, \tag{13}$$

and

$$F^{\pm}(\alpha_n) = 2\beta_n \left\{ \begin{array}{l} -\delta_{n0} \\ A_n\ . \end{array} \right. \tag{14}$$

The method is to solve the "+" equation for $N(x) = (ik_1)^{-1}N^{\mathrm{T}}(x)$ and evaluate the "−" equation for the A_n amplitudes. (The equations are formulated in terms of the scaled normal derivative). The CC solution for N^{T} is used in the "−" equation to evaluate the A_n. The kernel in Eq. (13) when discretized has row entries given by the discrete spectral index n and column entries sampled in coordinate space. The method is thus referred to as spectral-coordinate (SC).

The third formulation is derived as follows. Expand the boundary unknown in the topological or surface wave basis in Eq. (13)

$$N(x) = \sum_{j'=-\infty}^{\infty} \tilde{N}_{j'} e^{ik_1(\alpha_{j'}x - \beta_{j'}s(x))}. \tag{15}$$

The result is first a system of linear equations for the vector of expansion coefficients in the discrete spectral domain $\tilde{\mathbf{N}} = \{\tilde{N}_j\}$

$$\tilde{\mathbf{K}}\tilde{\mathbf{N}} = \mathbf{F}^+, \tag{16}$$

with the components

$$F_j^+ = -2\beta_0\delta_{j0}, \tag{17}$$

and the matrix whose entries are

$$\tilde{K}_{jj'} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(j-j')y} e^{ik_1(\beta_j - \beta_{j'})s(\frac{L}{2\pi}y)} dy, \tag{18}$$

where the integrals have been scaled to $[-\pi, \pi]$, and second, the set of equations to evaluate for the A_j coefficients is

$$\sum_{j'} \tilde{M}_{jj'}\tilde{N}_{j'} = 2\beta_j A_j, \tag{19}$$

where the matrix elements are

$$\tilde{M}_{jj'} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(j-j')y} e^{-ik_1(\beta_j + \beta_{j'})s(\frac{L}{2\pi})y} dy. \tag{20}$$

The scattered field is then given by Eq. (5). Rows and columns of both $\tilde{\mathbf{K}}$ and $\tilde{\mathbf{M}}$ are indexed in the (discrete) spectral integer j and the method is referred to as spectral-spectral (SS).

A detailed formulation of these three methods is available [2, 3] as is the extension of the methods to one-dimensional [4] and two-dimensional [5] transmission problems. Eq. (7) is the standard coordinate based method used for these problems [1].

3 Wavelet Analysis

The matrix system for the CC equations follows from the discretization of Eq. (7). It is written as

$$\mathbf{Z}\mathbf{N}^T = \mathbf{F}^i. \tag{21}$$

The matrix system for the SC equations follows from the discretization of the "+" Eq. (12). It is

$$\mathbf{K}\mathbf{N} = \mathbf{F}^+, \tag{22}$$

were F^+ is the vector of values over the index n.

The matrix system for the SS equations follows from the discussion in Eqs. (15)-(18):

$$\tilde{\mathbf{K}}\tilde{\mathbf{N}} = \mathbf{F}^{+}. \tag{23}$$

Each system thus has the matrix form

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{24}$$

for the vector of unknowns $\mathbf{x}$ where $\mathbf{A}$ and $\mathbf{b}$ are given. Details of these derivations can be found in [2, 3].

We apply wavelet transform methods to this system of equations in an attempt to speed up the process of solving for the vector $\mathbf{x}$. The goal is to sparsify the matrix which makes the matrix solution faster. Several different types of wavelets can be used. Among those we used are the Daubechies class of orthogonal wavelets [10]. The transform is performed using matrix multiplication and is a similarity transform in the first stage. If $\mathbf{W}$ is the orthogonal matrix which performs the wavelet transform on the columns of a matrix, then the wavelet transform $\tilde{\mathbf{A}}$ of the matrix $\mathbf{A}$ is given by

$$\tilde{\mathbf{A}} = \mathbf{W}\mathbf{A}\mathbf{W}^{T}, \tag{25}$$

since $\mathbf{W}^{-1} = \mathbf{W}^{T}$ (similarity transform). Applying this transformation to Eq. (24) we get

$$\mathbf{W}\mathbf{A}\mathbf{W}^{T}\mathbf{W}\mathbf{x} = \mathbf{W}\mathbf{b}, \tag{26}$$

and the system is just

$$\tilde{\mathbf{A}}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}, \tag{27}$$

where $\tilde{\mathbf{x}} = \mathbf{W}\mathbf{x}$ and $\tilde{\mathbf{b}} = \mathbf{W}\mathbf{b}$.

Pick a threshold τ and replace $\tilde{\mathbf{A}}$ by $\tilde{\mathbf{A}}^{(t)}$ as follows

$$\tilde{A}_{ij}^{(t)} = \begin{cases} 0, & |\tilde{A}_{ij}| \leq \tau \max_{m,n} |\tilde{A}_{mn}|, \\ \tilde{A}_{ij} & \text{otherwise.} \end{cases} \tag{28}$$

Solve the new system

$$\tilde{\mathbf{A}}^{(t)}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}, \tag{29}$$

for $\tilde{\mathbf{x}}$. Then compute $\mathbf{x} = \mathbf{W}^{T}\tilde{\mathbf{x}}$, which is the thresholded solution of Eq. (24) specific computational details can be found in [2, 3].

Several researchers have recently proposed the use of wavelets in the solution of matrix equations similar to Eq. (21) [11, 12, 13, 14, 15, 16]. Others have treated applications of wavelet methods to integral equations [17, 18, 19, 20, 21]. The technique we describe is related to filter theory and two extensive tutorial papers by Sarkar *et al* [22, 23] have recently appeared in this magazine. In addition, Sarkar and Kim [24] have applied wavelet-like transforms to large dense matrix systems. Our system of equations however, are comparatively small.

A comparison of our results with wavelet and thresholded wavelet methods to these other references is difficult since the latter are applied to canonical geometric shapes such as corner reflectors [12, 14], circular cylinders [11, 14], a thin wire and thin flat plate [13], a semicircular array of parallel thin cylinders [15], an elliptic cylinder [16], a rectangular cylinder [16], a pair of hemi-circular metal tubes [16], and dielectric waveguides [18]. Nominally the equations used in these references are similar to Eq. (7) with $\alpha = 0$, i.e. to our CC1 equations. But, because of the simple geometric shapes, the Green's functions have a much simpler form (only a single Hankel function) than ours. Further, ours involve a rough surface whose height and slope vary over a parametric set and contain other parameters such as angle of incidence and surface period. Our Green's functions are thus much more complicated and take a much longer time to evaluate. See the full discussion in [2, 3]. Nevertheless some brief points of agreement and disagreement are possible. First, the wavelet transform can be used to sparsify the impedance matrix and to make the resulting inversion more efficient in the sense that the inversion of just this matrix is faster. But this is not the full story. The times necessary to set up and undo the wavelet transform, and to do a matrix search to threshold can be many times the inversion time of the untransformed impedance matrix. This is particularly true for the SC and SS methods whose small matrix sizes make the inversion much faster. We discussed this in a previous report [25]. Second, these other references quite often reach a sparsity of 10 % or so (only 10 % of the matrix elements remain after thresholding). We could not reach these levels of sparsity without our matrices becoming singular unless we dramatically oversampled to begin with. A discussion is in the next section. In addition it is claimed in all the references that very sparse matrices did not significantly affect the accuracy. No references had the energy check levels we obtained. (The method of moments solution was assumed to be the exact result). We illustrate in our conclusions that the accuracy decreased dramatically as sparsity increased. Note that this conclusion is based on the fact that we computed a matrix inverse whereas the references used some iterative technique. Sparsification may help iterative techniques.

It should be remembered that we have control over sampling. Proponents of wavelets go immediately to large matrices which are often the result of oversampling. Significant sparsification may not appreciably alter the accuracy. We on the other hand often achieved better results with small dense matrices and no wavelet techniques. The specific conclusions follow.

4 Computational Results

This section presents timing and reliability results for several formalisms on several surfaces. $\lambda/\Delta x$ is a measure of pulse width, with higher numbers corresponding to faster sampling. $\lambda/\Delta x = 10$ is often quoted, but it can be either oversampling or undersampling. The best CC method is used for the surface current and scattered amplitude plots, since spectral methods often have incorrect currents. An additional result is referred to as CG, a CC Galerkin approach with Fourier basis functions for a first kind equation. This is very similar to CC1, but the self-cell integral is performed exactly, not using the first few terms in an expansion.

A detailed discussion of the numerical procedure for CC, SC, CG and SS methods can be found in [2, 3]. For our calculations with CG, we set $\alpha = 0$ and $\beta = 1$.

We treat three rough surface examples. All examples are for the simple periodic surface $s(x) = -(d/2)\cos(2\pi x/L)$. For Example 1 we have $d/L = 0.15, \lambda/L = 0.007818$ (256 Bragg modes) and $\theta^{\mathrm{i}} = 20°$ (close to normal incidence). The maximum slope is $\pi d/L$. For Example 2, $d/L = 2.5, \lambda/L = 100$ (only one Bragg mode in the specular direction) and $\theta^{\mathrm{i}} = 20°$ (close to normal incidence). For Example 3, $d/L = 0.06, \lambda/L = 0.007799$ (256 Bragg modes) and $\theta^{\mathrm{i}} = 80°$ (near grazing incidence).

In Tables 1-6 we compare the five different methods SC, SS, CC1, CC2 and CG. The matrix fill time for SC was always the smallest since we are only evaluating a simple function to fill the matrix. For SS we must evaluate the integral, Eq. (18). The coordinate based methods all involve the evaluation of the periodic Green's function which is very time consuming. The "Error" was given by

$$\text{Error} = \log_{10} |1 - \text{Normalized Energy}|, \tag{30}$$

where the Normalized Energy was computed from the left hand side of Eq. (6). (The Error functional was not used in a minimization sense but only as an a fortiori check. It is a necessary but not sufficient check on the accuracy). For Examples 1 and 2 (Tables 1-4) extremely high levels of accuracy where achieved. For Example 3 (Tables 5 and 6) it was necessary to go to larger size matrices to achieve this high level of accuracy. Solution times are compared for the solution without any wavelets and with three different thresholded wavelet methods. ($\tau = 0$, i.e. no thresholding, $\tau = 0.001$ and $\tau = 0.01$) with corresponding sparsity Sp indicated (fraction of zero elements). "Flops" refers to $\log_{10}$ of the number of floating point operations. Computations were done using a SUN SPARC 2.0 workstation and the software was MATLAB 4.2. Times are in seconds of CPU time.

From Tables 1, 3 and 5 where no wavelet methods were used, we can draw several conclusions. Clearly the SC method was the preferred method with an overwhelming advantage in terms of fill time. Because of its partly spectral nature it already contains the topological basis functions. However, SC failed to work for very rough and highly oscillatory surfaces [2, 3]. The SS method additionally contains the topological basis in the expansion of the boundary unknown in Eq. (15) and it had similar convergence difficulties. The coordinate based methods (CC1, CC2, and CG1) did not have these convergence difficulties, but it was almost always necessary to go to sufficiently large matrices before the error was small. This added considerably to fill time and solution time. Matrix size for the spectral based methods was closely tied to the number of real Bragg modes. Increases in the matrix size required the addition of evanescent or growing "modes" and this was not always advantageous.

From Tables 2, 4 and 6 where wavelet methods were used we have several conclusions. The overall time required to solve these size matrices was always longer with our wavelet code. The accuracy worsens as sparsity increases. Note that we solve our problems using row reduction so that a high level of sparsity can produce a singular matrix. (We contrast our results with iterative techniques where high sparsity is generally advantageous). Picking a threshold yielding a nonsingular matrix with certain sparsity and/or accuracy can be very difficult. Note that we control sampling. If one doesn't mind the accuracy drop obtained with wavelets, one should have been using a smaller matrix initially.

The relation of error to sparsity is further illustrated in Figs. 2-4 for the three examples using CC2. The plots stop when the matrices become singular. This fall-off in accuracy with sparsity is often very dramatic. Note that in Fig. 4 the 256×256 matrix at 75 % sparsity has an energy balance inaccuracy almost seven orders of magnitude worse than the dense 128×128 matrix (at 0 % sparsity) even though both have the same number of non-zero entries.

Finally, we note that we have restricted our problem to small matrices ($\leq 512 \times 512$), and have used only the discrete wavelet transform with the Daubechies filter with 6 filter coefficients (DAUB6). Further, we have methods which are highly accurate. The accuracy on our problems is far in excess of other references which apply wavelets to canonical shapes. We are also describing non-iterative solution methods and we recognize that different conclusions may occur for iterative techniques. Times increase when wavelets are used anywhere however and we believe that the significant decrease of accuracy with sparsity is a matter which has not been thoroughly addressed before in the literature.

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Type	Matrix Size	Fill Time	$\lambda/\Delta x$	*Error*	Time	Flops
SC	256	2.2	2.0	-6.1	4.2	7.67
	266	2.2	2.1	-12.1	4.7	7.72
	276	2.4	2.2	-6.9	5.2	7.77
SS	256	27000		-15.4	1.6	7.37
	266	30000		-12.1	4.3	7.72
	276	33000		-7.3	4.7	7.77
CC1	128	1900	1.0	-0.2	0.6	6.79
	256	7400	2.0	-1.0	3.9	7.67
	512	30000	4.0	-2.7	29	8.56
CC2	128	2600	1.0	1.6	0.6	6.79
	256	11000	2.0	-4.6	3.9	7.67
	512	43000	4.0	-8.0	28	8.56
CG1	128	1800	1.0	-0.2	0.6	6.79
	256	7300	2.0	-0.9	3.9	7.67
	512	29000	4.0	-3.3	28	8.56

Table 1: Results for Example 1: $s(x) = -(d/2)\cos(2\pi x/L)$, $d/L = 0.15$, $\lambda/L = 0.007818$, and $\theta^i = 20°$. No wavelet techniques were used here.

	Matrix	$\tau = 0$			$\tau = 0.001$				$\tau = 0.01$			
Type	Size	*Error*	Time	Flops	*Error*	Sp	Time	Flops	*Error*	Sp	Time	Flops
CC1	128	-0.2	4.4	7.11	-0.2	0.175	4.9	7.10	1.5	0.790	4.0	6.90
	256	-1.0	20	7.90	-1.1	0.545	22	7.79	*	0.652	*	*
	512	-2.7	114	8.71	-2.5	0.333	137	8.61	*	0.613	*	*
CC2	128	1.6	4.1	7.11	1.8	0.100	4.8	7.10	2.4	0.774	4.1	6.91
	256	-4.6	20	7.90	-1.3	0.557	21	7.73	*	0.665	*	*
	512	-8.0	112	8.71	-0.02	0.394	140	8.65	*	0.842	*	*
CG1	128	-0.2	4.1	7.11	-0.2	0.151	4.7	7.10	3.2	0.539	4.3	7.01
	256	-0.9	19	7.90	-0.9	0.181	23	7.87	2.8	0.615	21	7.78
	512	-3.3	110	8.71	-3.3	0.185	141	8.68	-0.5	0.582	136	8.66

Table 2: Results for Example 1: $s(x) = -(d/2)\cos(2\pi x/L)$, $d/L = 0.15$, $\lambda/L = 0.007818$, and $\theta^i = 20°$. Solutions included wavelet transforms with thresholds given by τ and sparsity Sp (fraction of zero elements). A * denotes that the thresholded matrix was singular.

Type	Matrix Size	Fill Time	$\lambda/\Delta x$	*Error*	Time	Flops
SC	1	.02	100	-15.7	0.02	0.85
	5	.02	500	-15.7	0.02	3.16
	11	.04	1100	-15.4	0.02	3.91
SS	1	.4		-2.0	0.01	0.30
	11	12		-7.5	0.01	3.88
	21	46		-8.2	0.01	4.59
CC1	64	160	6400	-15.4	0.1	5.92
	128	560	12800	-15.4	0.6	6.79
	256	2200	25600	-15.1	4.1	7.67
CC2	64	280	6400	-6.7	0.1	5.92
	128	931	12800	-8.7	0.6	6.79
	256	3500	25600	-9.0	3.9	7.67
CG1	64	150	6400	-15.2	0.1	5.92
	128	510	12800	-15.7	0.6	6.79
	256	2000	25600	-15.7	3.9	7.67

Table 3: Results for Example 2: $s(x) = -(d/2)\cos(2\pi x/L)$, $d/L = 2.5$, $\lambda/L = 100$, $\theta^i = 20°$. No wavelet techniques were used here.

Type	Matrix Size	$\tau = 0$			$\tau = 0.001$				$\tau = 0.01$			
		Error	Time	Flops	*Error*	Sp	Time	Flops	*Error*	Sp	Time	Flops
CC1	64	-14.6	1.0	6.35	-14.6	0.000	1.1	6.35	-14.6	0.000	1.1	6.35
	128	-15.1	4.2	7.11	-15.1	0.000	4.0	7.11	11.1	0.500	4.7	7.10
	256	-14.5	20	7.90	-14.5	0.000	19	7.90	14.7	0.813	25	7.89
CC2	64	-6.7	1.1	6.35	-5.5	0.205	1.2	6.32	*	0.689	*	*
	128	-8.7	3.9	7.11	-5.0	0.450	4.5	7.01	*	0.829	*	*
	256	-9.0	20	7.90	-2.9	0.716	19	7.65	*	0.909	*	*
CG1	64	-15.0	1.0	6.35	*	0.608	*	*	*	0.757	*	*
	128	-15.4	4.0	7.11	*	0.807	*	*	*	0.856	*	*
	256	-14.9	19	7.90	*	0.900	*	*	*	0.922	*	*

Table 4: Results for Example 2: $s(x) = -(d/2)\cos(2\pi x/L)$, $d/L = 2.5$, $\lambda/L = 100$, $\theta^i = 20°$. Solutions included wavelet transforms with thresholds given by τ and sparsity Sp (fraction of zero elements). A * denotes that the thresholded matrix was singular.

Type	Matrix Size	Fill Time	$\lambda/\Delta x$	*Error*	Time	Flops
SC	256	1.9	2.0	-1.2	3.8	7.67
	272	2.2	2.1	-0.9	4.6	7.74
	288	2.4	2.2	-4.9	5.4	7.82
SS	256	20000		-0.6	1.6	7.37
	272	25000		-1.8	4.6	7.75
	288	30000		-1.6	5.3	7.82
CC1	128	2800	1.0	-0.4	0.6	6.79
	256	11000	2.0	-0.3	3.9	7.67
	512	45000	4.0	-5.3	28	8.56
CC2	128	4300	1.0	-0.2	0.6	6.79
	256	17000	2.0	-0.6	3.9	7.67
	512	69000	4.0	-4.8	28	8.56
CG1	128	2900	1.0	-0.3	.6	6.79
	256	12000	2.0	-0.3	4.1	7.67
	512	45000	4.0	-4.2	29	8.56

Table 5: Results for Example 3: $s(x) = -(d/2)\cos(2\pi x/L)$, $d/L = 0.06$, $\lambda/L = 0.007799$, and $\theta^i = 80°$. No wavelet techniques were used here.

Type	Matrix Size	$\tau = 0$			$\tau = 0.001$				$\tau = 0.01$			
		Error	Time	Flops	*Error*	Sp	Time	Flops	*Error*	Sp	Time	Flops
CC1	128	-0.4	4.1	7.11	-0.4	0.620	4.1	6.95	0.2	0.889	3.9	6.89
	256	-0.3	20	7.90	-0.3	0.555	22	7.79	*	0.660	*	*
	512	-5.3	113	8.71	-2.5	0.344	135	8.62	*	0.639	*	*
CC2	128	-0.2	4.0	7.11	-0.2	0.275	4.7	7.05	0.9	0.823	4.3	6.97
	256	-0.6	20	7.90	-0.6	0.567	21	7.76	*	0.682	*	*
	512	-4.8	115	8.71	-1.4	0.526	134	8.59	*	0.908	*	*
CG1	128	-0.3	3.9	7.11	-0.3	0.255	4.5	7.08	*	0.806	*	*
	256	-0.3	19	7.90	-0.3	0.209	23	7.88	3.0	0.656	21	7.75
	512	-4.2	107	8.71	-3.6	0.241	137	8.67	2.4	0.680	118	8.53

Table 6: Results for Example 3: $s(x) = -(d/2)\cos(2\pi x/L)$, $d/L = 0.06$, $\lambda/L = 0.007799$, and $\theta^i = 80°$. Solutions included wavelet transforms with thresholds given by τ and sparsity Sp (fraction of zero elements). A * denotes that the thresholded matrix was singular.