SYMBOLIC SOFTWARE FOR THE STUDY OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Three entirely symbolic MACSYMA programs are presented. The first one carries out the Painlevé integrability test, the second computes solitons and the third program aids in finding the Lie symmetry group for systems of differential equations.

1 Introduction

Apart from various packages for the calculation of Lie symmetries, symbolic software for nonlinear partial differential equations (PDEs) is virtually nonexistent.

Three symbolic programs will be presented. The first one automatically carries out the Painlevé integrability test for a single PDE with polynomial terms. There is no limitation on the order or the degree of nonlinearity. Variable coefficients are permitted in the equation.

The second program tests for the existence of exact solitary wave and soliton solutions of nonlinear PDEs. It also explicitly constructs multi-soliton solutions via Hirota's direct method.

The third symbolic program fully implements the algorithm for the calculation of Lie point symmetries. It is designed to handle large and complicated systems of differential equations.

The three programs are written in MACSYMA syntax, they are entirely symbolic and produce exact analytical output. The programs require little experience with MACSYMA and minimal interaction on the part of the users.

The goal is to provide high quality symbolic programs to researchers working on topics such as soliton theory, dynamical systems, and more generally on wave phenomena in meteorology, bio-sciences, fluid dynamics, plasma and particle physics, and nonlinear optics.

2 The Painlevé Test

A PDE is said to possess the Painlevé property [1] if its solutions in the complex plane are single-valued in the neighborhood of non-characteristic, movable singular manifolds. For ODEs, this comes down to saying that solutions should have no worse singularities than movable poles. Such equations are prime candidates for being completely integrable, in the sense that they are linearizable via the Inverse Scattering Transform.

For instance, one could ask for which time-dependent coefficients $a(t)$ and $b(t)$ the equation [14]

$$(u_t + 6uu_x + u_{xxx})_x + a(t)u_x + b(t)u_{yy} = 0 \tag{1}$$

will posses the Painlevé property. The Painlevé test [1], which is entirely algorithmic, is performed automatically with the MACSYMA program PAINLEVE.MAX [5, 6].

Seeking a solution of the form

$$u(x, y, t) = g(x, y, t)^{\alpha} \sum_{k=0}^{\infty} u_k(x, y, t) g(x, y, t)^k, \tag{2}$$

where the singularity manifold is denoted by $g(x, y, t)$, the program determines that $\alpha = -2$ and $u_0 = -2(g_x)^2$ by substituting $u \sim u_0 g^{\alpha}$ into the equation. Next, the program determines for which values of k arbitrary coefficients u_k will arise in (2). Technically, this is done by substituting $u \sim u_0 g^{\alpha} + u_r g^{r+\alpha}$ into the equation and by requiring that u_r be arbitrary. The special values for r for which this happens are called the resonances. For (1) the resonances are $r = 4, 5$ and 6. Since the equation is nonlinear, at every resonance level a compatibility condition must be satisfied. For (1), the compatibility conditions at levels 4 and 5 are identically satisfied. At resonance level $r = 6$ the compatibility condition

$$[a_t + 2a^2]{g_x}^3 + [b_t + 4ab]({g_x}^2 g_{yy} + g_{xx}{g_y}^2 - 2g_x g_{xy} g_y) = 0, \tag{3}$$

must hold irrespective the form of the singularity manifold $g(x, y, t)$. It is clear that (1) will have the Painlevé property if $a_t + 2a^2 = 0$, $b_t + 4ab = 0$. Two cases are possible. The first case $a(t) = 0, b(t) = c$, for any constant c, leads to the ubiquitous Korteweg-de Vries (KdV) equation ($c = 0$), or to the Kadomtsev-Petviashvili (KP) equation ($c = \pm 1$). The more interesting second case, with $a(t) = 1/[2(t - t_0)]$, $b(t) = k/(t - t_0)^2$, where k and t_0 are integration constants, leads for $k = 0$ and upon integration with respect to x, to the cylindrical KdV equation [1],

$$u_t + 6uu_x + u_{xxx} + \frac{u}{2t} = 0, \tag{4}$$

which describes solitary waves in a channel with slowly varying depth. For $k \neq 0$, upon appropriate scaling of the variable y, equation (1) reduces to the well-known cylindrical Kadomtsev-Petviashvili equation [1],

$$(u_t + 6uu_x + u_{xxx})_x + \frac{1}{2t}u_x + \frac{3\sigma^2}{t^2} y_{yy} = 0, \qquad \sigma^2 = \pm 1, \tag{5}$$

which was derived [10] for surface waves in a fluid which are characterized by small deviation from axial symmetry. The KdV and KP equations and their cylindrical generalizations (4) and (5) are all known to be completely integrable [1, 4].

Next, consider the generalized KP equation with variable coefficients,

$$(u_t + uu_x + u_{xxx})_x + a(y,t)u_x + b(y,t)u_y + c(y,t)u_{yy} + d(y,t)u_{xy} + e(y,t)u_{xx} = 0, \quad (6)$$

which was first tested for the Painlevé property by Ablowitz and Clarkson [1]. For an equation as complicated as (6) it pays off to write the non-characteristic manifold as $g(x,y,t) = x - h(y,t)$, where $h(y,t)$ is analytic in its arguments. We adhere to the notation used in the program PAINLEVE.MAX, where upon suggestion of Kruskal this optional simplification is implemented. The Painlevé analysis of (6) is very similar to that of (1) and again only one compatibility condition occurs at resonance level $r = 6$,

$$\begin{aligned} &[a_t + 2a^2 + da_y - be_y - ce_{yy}] - [b_t + 4ab - bd_y - cd_{yy} + 2ca_y + db_y]h_y \\ &- [c_t + 4ac - 2cd_y + dc_y]h_{yy} + [2b^2 - bc_y - cc_{yy} + 2cb_y]{h_y}^2 + 2c[2b - c_y]h_y h_{yy} = 0. \quad (7)\end{aligned}$$

The coefficients of the separate terms in derivatives of h must vanish identically. After integration [1] of the five resulting equations two cases emerge. The first one leads to (1) with $b = 0$ for which the Painlevé analysis was given above. The second case leads to an equation which can be transformed into the standard KP equation, this is (5) without the $u_x/(2t)$ term.

As a final example, consider the KdV equation with time dependent coefficients,

$$u_t + a(t)uu_x + b(t)u_{xxx} = 0. \quad (8)$$

With the program we found that (8) possesses the Painleve property provided

$$a^2 b_{tt} - 3aa_t b_t + (3{a_t}^2 - aa_{tt})b = 0. \quad (9)$$

Introducing the auxiliary variable $v(t) = b(t)/a(t)$, this differential equation can be written as $v_{tt} - \frac{a_t}{a} v_t = 0$, which is readily integrated. One retrieves Joshi's result [11],

$$v(t) = b(t)/a(t) = c_1 \int^t a(\tau)d\tau + c_2, \quad (10)$$

where c_1, c_2 are integration constants.
If we select $a(t) = t^n$, with n integer but $n \neq -1$, then $b(t) = k_1 t^{2n+1} + k_2 t^n$ and (8) reduces to

$$u_t + t^n uu_x + (k_1 t^{2n+1} + k_2 t^n)u_{xxx} = 0, \quad (11)$$

where k_1 and k_2 are completely arbitrary constants.

Skillful use of the Painlevé program in connection with a Laurent expansion (2) truncated at the constant level term, allows to quickly find exact closed form solutions of nonlinear PDEs. For instance, consider the famous FitzHugh-Nagumo (FHN) equation [2],

$$u_t - u_{xx} + u(1-u)(a-u) = 0, \quad (12)$$

where $-1 \leq a < 1$ is a constant parameter. To determine the velocities of traveling wave solutions we apply the Painlevé test to the ODE,

$$c\,\phi_z + \sqrt{2}\,\phi_{zz} - \sqrt{2}\,\phi\,(1-\sqrt{2}\,\phi)(a-\sqrt{2}\,\phi) = 0, \tag{13}$$

where $\phi(z) = \phi(x - \frac{ct}{\sqrt{2}}) = \frac{1}{\sqrt{2}}\,u(x,t)$. Equation (13) passes the Painlevé test if

$$c(c-2a+1)(c+a-2)(c+a+1) = 0. \tag{14}$$

It turn out that for each of these c values an exact solution of (12) can be constructed. Using a Laurent series of type (2) but truncated at the constant level term,

$$u(x,t) = g(x,t)^{-1}[u_0(x,t) + u_1(x,t)g(x,t)] = \sqrt{2}\,\frac{g_x(x,t)}{g(x,t)} + u_1(x,t), \tag{15}$$

and setting $u_1(x,t) = 0$ for simplicity, one gets

$$g_t - 3g_{xx} + \sqrt{2}(1+a)\,g_x \;=\; 0, \tag{16}$$

$$g_{tx} - g_{xxx} + a\,g_x \;=\; 0, \tag{17}$$

by substituting (15) into (12) and equating power terms in g. The above equations are easily integrated:

$$g(x,t) = A\,exp[\frac{1}{\sqrt{2}}(x - \frac{1}{\sqrt{2}}(2a-1)t)] + B\,exp[\frac{a}{\sqrt{2}}(x - \frac{1}{\sqrt{2}}(2-a)t)] + C, \tag{18}$$

where A, B and C are integration constants. Substitution of (18) into (15) then gives

$$u(x,t) = \frac{A\,exp[\frac{1}{\sqrt{2}}(x-\frac{1}{\sqrt{2}}(2a-1)t)] + aB\,exp[\frac{a}{\sqrt{2}}(x-\frac{1}{\sqrt{2}}(2-a)t)]}{A\,exp[\frac{1}{\sqrt{2}}(x-\frac{1}{\sqrt{2}}(2a-1)t)] + B\,exp[\frac{a}{\sqrt{2}}(x-\frac{1}{\sqrt{2}}(2-a)t)] + C}. \tag{19}$$

This exact solution, which describing the coalescence of two wave fronts, can also be obtained with Hirota's bilinear method [12] or via a nonclassical method of symmetry reduction [13]. Single solitary wave solutions follow as special cases of (19). For example, with $A = 0$ one has

$$u(x,t) = \frac{a}{2}(1 + \tanh[\frac{a}{2\sqrt{2}}(x - \frac{(2-a)}{\sqrt{2}}t) + \delta]), \tag{20}$$

where $\delta = (1/2)\ln(B/C)$ is the arbitrary constant phase. This is the traveling wave solution for $c = 2 - a$ in (14).

The program PAINLEVE.MAX provides a valuable tool to investigate possible integrability of differential equations. As the above examples show, application of the program to equations with variable coefficients allows to automatically calculate the necessary conditions for the coefficients in the equations to make them prime candidates for integrability. A write-up of the program PAINLEVE.MAX and more examples with their actual MACSYMA output are given in [5].

3 Soliton Solutions via Hirota's Method

The programs HIROTA_SINGLE.MAX [7, 15] and HIROTA_SYSTEM.MAX [15] allow to investigate the existence of multi-soliton solutions and to obtain their explicit forms. Thus far the programs, which are based on an algorithm due to Hirota [3], can handle equations of KdV type and modified KdV type and calculate their one, two and three soliton solutions explicitly.

As a first example, consider the Sawada-Kotera equation,

$$u_t + 45u^2 u_x + 15u_x u_{2x} + 15uu_{3x} + u_{5x} = 0. \tag{21}$$

This equation, which is of KdV type, can be written in bilinear form as

$$(D_x D_t + D_x^6)(f \cdot f) = 0, \tag{22}$$

with

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}, \tag{23}$$

and where D_x, D_t denote Hirota's operators. These operators are defined [3] by

$$D_x^m\, D_t^n (f \cdot g) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(x,t)\, g(x',t')\Big|_{x'=x,t'=t}, \tag{24}$$

with non-negative integers m and n. For the three soliton solution the program HIROTA_SINGLE.MAX determines

$$\begin{aligned} f(x,t) \quad = \quad & 1 + \exp\theta_1 + \exp\theta_2 + \exp\theta_3 + b_{123}\ \exp(\theta_1 + \theta_2 + \theta_3) \\ & + a_{12}\ \exp(\theta_1 + \theta_2) + a_{13}\ \exp(\theta_1 + \theta_3) + a_{23}\ \exp(\theta_2 + \theta_3), \end{aligned} \tag{25}$$

where $\theta_i = k_i x - \omega_i t + \delta_i$, and with the dispersion relation $\omega = k^5$. The coefficients are computed in factored form as

$$a_{ij} = \frac{(k_i - k_j)^2\,(k_i^2 - k_i k_j + k_j^2)}{(k_i + k_j)^2 (k_i^2 + k_i k_j + k_j^2)},\ \ i,j = 1,2,3 \text{ and } i < j, \tag{26}$$

$$\begin{aligned} b_{123} &= a_{12}\, a_{13}\, a_{23} \\ &= \frac{(k_1 - k_2)^3\,(k_1^3 + k_2^3)(k_1 - k_3)^3\,(k_1^3 + k_3^3)(k_2 - k_3)^3\,(k_2^3 + k_3^3)}{(k_1 + k_2)^3\,(k_1^3 - k_2^3)(k_1 + k_3)^3\,(k_1^3 - k_3^3)(k_2 + k_3)^3\,(k_2^3 - k_3^3)}. \end{aligned} \tag{27}$$

The program confirms the existence of a four soliton solution.

As a second example we take the bilinear equation [8]

$$(kD_t^2 + D_x^3 D_t + D_x^6)(f \cdot f) = 0, \tag{28}$$

with an arbitary parameter k. It is known that this equation passes the Painlevé test for $k = -1/5$. With the program HIROTA_SINGLE.MAX we calculate the condition for the existence of a three soliton solution:

$$-\frac{972}{k^3}(5k+1)(2k^2+2k+\sqrt{1-4k}-1)k_1^4k_2^4k_3^4(k_2-k_1)^2(k_2+k_1)^2(k_3-k_1)^2(k_3+k_1)^2(k_3-k_2)^2(k_3+k_2)^2=0, \tag{29}$$

fueling the conjecture that the existence of a three soliton solution implies complete integrability. With $k = -1/5$ we continue the calculation of the two and three soliton solutions of (28). We obtained completely automatically

$$a_{ij} = \frac{(k_i - k_j)^2 (\sqrt{5}k_i^2 + 3k_i^2 + \sqrt{5}k_ik_j + k_ik_j + \sqrt{5}k_j^2 + 3k_j^2)}{(k_i + k_j)^2(\sqrt{5}k_i^2 + 3k_i^2 + \sqrt{5}k_ik_j + k_ik_j + \sqrt{5}k_j^2 + 3k_j^2)}, \quad i, j{=}1, 2, 3 \text{ and } i < j, \tag{30}$$

$$b_{123} = a_{12}\, a_{13}\, a_{23}. \tag{31}$$

Due to space limitations of MACSYMA on a VAX 8600, we have not been able to verify the existence of a four soliton solution of (28).

As a last example, consider the modified KdV equation,

$$u_t + 6u^2u_x + u_{3x} = 0, \tag{32}$$

which cannot be transformed into a single bilinear equation. However, by a suitable change of variables, $u = w_x = \ln(f/g)$, equation (32) can be written as a system of two coupled bilinear equations, namely,

$$(D_x^3 + D_t)(f \cdot g) = 0, \tag{33}$$

$$D_x^2(f \cdot g) = 0. \tag{34}$$

Replacing the term D_x^3 by D_x^{2p+1} leads to a family of higher-order modified KdV equations [9]. For $p = 1, 2$ and 3, the program HIROTA_SYSTEM.MAX confirms that two, three and four soliton solutions exist. However, for $p = 4$ there is no longer a three soliton solution. For all the cases, the two soliton solution is generated by

$$f(x,t) = 1 + i\exp\theta_1 + i\exp\theta_2 - a_{12}\,\exp(\theta_1 + \theta_2)\,, \tag{35}$$

$$g(x,t) = 1 - i\exp\theta_1 - i\exp\theta_2 - a_{12}\,\exp(\theta_1 + \theta_2)\,, \tag{36}$$

and the three soliton solution (for $p = 1, 2, 3$) by

$$f(x,t) = 1 + i\exp\theta_1 + i\exp\theta_2 + i\exp\theta_3 - ib_{123}\,\exp(\theta_1 + \theta_2 + \theta_3) - a_{12}\,\exp(\theta_1 + \theta_2) - a_{13}\,\exp(\theta_1 + \theta_3) - a_{23}\,\exp(\theta_2 + \theta_3), \tag{37}$$

$$g(x,t) = 1 - i\exp\theta_1 - i\exp\theta_2 - i\exp\theta_3 + i\,b_{123}\,\exp(\theta_1 + \theta_2 + \theta_3) - a_{12}\,\exp(\theta_1 + \theta_2) - a_{13}\,\exp(\theta_1 + \theta_3) - a_{23}\,\exp(\theta_2 + \theta_3). \tag{38}$$

For the modified KdV equation and its higher-order generalizations, the coefficients are

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad i, j = 1, 2, 3 \quad \text{and} \quad i < j, \tag{39}$$

$$b_{123} = a_{12}\, a_{13}\, a_{23} = \frac{(k_1 - k_2)^2\,(k_1 - k_3)^2\,(k_2 - k_3)^2}{(k_1 + k_2)^2\,(k_1 + k_3)^2\,(k_2 + k_3)^2}. \tag{40}$$

Many more examples and the code of the programs HIROTA_SINGLE.MAX and HIROTA_SYSTEM.MAX can be found in [15].

4 Lie Point Symmetries

The program SYMMGRP.MAX [2] automatically calculates the determining equations for the coefficients in the vector field that realizes the Lie algebra of point symmetries. With a feedback mechanism, these determining equations can then be solved explicitly. The flow corresponding to the infinitesimal generators can be obtained via simple integration.

As an example, consider the Harry Dym equation [1]

$$u_t - u^3 u_{xxx} = 0. \tag{41}$$

Clearly, this is one equation ($m = 1$) with two independent variables ($p = 2$) and one dependent variable ($q = 1$). We denote the variables by $x[1] = x,\ x[2] = t$ and $u[1] = u$. The equation is then entered in the form $e1 = u[1,[0,1]] - u[1]^3 * u[1,[3,0]]$, and the variable $v1 = u[1,[0,1]]$ is selected for elimination.

The program SYMMGRP.MAX automatically computes the determining equations for the coefficients eta[1] $= \eta^x$, eta[2] $= \eta^t$ and phi[1] $= \varphi^u$ of the vector field

$$\alpha = \eta^x \frac{\partial}{\partial x} + \eta^t \frac{\partial}{\partial t} + \varphi^u \frac{\partial}{\partial u}. \tag{42}$$

There are only eight determining equations,

$$\frac{\partial \text{eta2}}{\partial u[1]} = 0, \qquad \frac{\partial \text{eta2}}{\partial x[1]} = 0, \qquad \frac{\partial \text{eta1}}{\partial u[1]} = 0, \qquad \frac{\partial^2 \text{phi1}}{\partial u[1]^2} = 0,$$

$$\frac{\partial^2 \text{phi1}}{\partial u[1] \partial x[1]} - \frac{\partial^2 \text{eta1}}{\partial x[1]^2} = 0, \qquad \frac{\partial \text{phi1}}{\partial x[2]} - u[1]^3\, \frac{\partial^3 \text{phi1}}{\partial x[1]^3} = 0,$$

$$3u[1]^3\, \frac{\partial^3 \text{phi1}}{\partial u[1] \partial x[1]^2} + \frac{\partial \text{eta1}}{\partial x[2]} - u[1]^3\, \frac{\partial^3 \text{eta1}}{\partial x[1]^3} = 0, \qquad u[1]\, \frac{\partial \text{eta2}}{\partial x[2]} - 3u[1]\, \frac{\partial \text{eta1}}{\partial x[1]} + 3\,\text{phi1} = 0. \tag{43}$$

The general solution, written in the original variables, is

$$\begin{aligned} \eta^x &= k_1 + k_3\, x + k_5\, x^2, \\ \eta^t &= k_2 - 3k_4\, t, \\ \varphi^u &= (k_3 + k_4 + 2k_5\, x)\, u, \end{aligned} \tag{44}$$

and the five infinitesimal generators are

$$\begin{aligned} G_1 &= \partial_x\ , & G_2 = \partial_t\ , \\ G_3 &= x\partial_x + u\partial_u\ , & G_4 = -3t\partial_t + u\partial_u\ , \\ G_5 &= x^2\partial_x + 2xu\partial_u\ . \end{aligned} \tag{45}$$

Clearly, (41) is invariant under translations (G_1 and G_2) and scaling (G_3 and G_4). Computation of the flow corresponding to G_5 requires integration of the system

$$\begin{aligned} \frac{d\tilde{x}}{d\epsilon} &= \tilde{x}^2, & \tilde{x}(0) &= x, \\ \frac{d\tilde{t}}{d\epsilon} &= 0, & \tilde{t}(0) &= t, \\ \frac{d\tilde{u}}{d\epsilon} &= 2\tilde{x}\tilde{u}, & \tilde{u}(0) &= u, \end{aligned} \tag{46}$$

where ϵ is the parameter of the transformation group. One readily obtains $\tilde{x}(\epsilon) = x/(1-\epsilon x)$, $\tilde{t}(\epsilon) = t$, and $\tilde{u}(\epsilon) = u/(1-\epsilon x)^2$. We therefore conclude that for any solution $u = f(x,t)$ of equation (41) the transformed solution $\tilde{u}(\tilde{x},\tilde{t}) = (1+\epsilon\tilde{x})^2 f(\frac{\tilde{x}}{1+\epsilon\tilde{x}}, \tilde{t})$ will solve $\tilde{u}_{\tilde{t}} - \tilde{u}^3\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0$. A worked example of the calculation of Lie point symmetries of a system of PDEs is given in [2]. That paper also has a complete write-up of the program SYMMGRP.MAX and an extensive review of other symbolic packages for Lie symmetries.

References

[1] M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge: Cambridge University Press, 1991.

[2] B. Champagne, W. Hereman and P. Winternitz, “The computer calculation of Lie point symmetries of large systems of differential equations”, Comp. Phys. Comm., Vol. 66, pp. 319-340, 1991.

[3] P.G. Drazin and R.S. Johnson, Solitons: An Introduction, Cambridge: Cambridge University Press, 1989.

[4] V. S. Dryuma, “On the integration of the cylindrical Kadomtsev-Petviashvili equation by the method of the inverse problem of scattering theory”, Sov. Math. Dokl., Vol. 27, pp. 6-8, 1983.

[5] W. Hereman and S. Angenent, "The Painlevé test of nonlinear ordinary and partial differential equations", MACSYMA Newsletter, Vol. 6, pp. 11-18, 1990.

[6] W. Hereman and E. Van den Bulck, "MACSYMA program for the Painlevé test of nonlinear ordinary and partial differential equations", in: Proceedings of the Workshop on Finite Dimensional Integrable Nonlinear Dynamical Systems, eds: P. G. L. Leach and W.-H. Steeb, Singapore: World Scientific, 1988, pp. 117-129.

[7] W. Hereman and W. Zhuang, "Symbolic computation of solitons with MACSYMA", in: Proceedings of the 13th IMACS World Congress, Dublin, Ireland, 1991, eds.: W. F. Ames and P. J. van der Houwen. Amsterdam: Elsevier, 1992.

[8] J. Hietarinta, "A search for bilinear equations passing Hirota's three-soliton condition, Part I. KdV-type bilinear equations", J. Math. Phys., Vol. 28, pp. 1732-1742, 1987.

[9] M. Ito, "An extension of nonlinear evolution equations of the K-dV (mK-dV) type of higher orders",
J. Phys. Soc. Jpn., Vol. 49, pp. 771-778, 1980.

[10] R. S. Johnson, "Water waves and Korteweg-de Vries equations", J. Fluid Mech., Vol. 97, pp. 701-719, 1980.

[11] N. Joshi, "Painlevé property of general variable-coefficient versions of the Korteweg-de Vries and non-linear Schrödinger equations", Phys. Lett. A, Vol. 125, pp. 456-460, 1987.

[12] T. Kawahara and M. Tanaka, "Interactions of traveling fronts: An exact solution of a nonlinear diffusion equation", Phys. Lett. A, Vol. 97, pp. 311-314, 1983.

[13] M. C. Nucci and P. A. Clarkson, "The nonclassical method is more general than the direct method for symmetry reductions. An example of the Fitzhugh-Nagumo equation", Phys. Lett. A, Vol. 164, pp. 49-56, 1992

[14] W.-H. Steeb and N. Euler, Nonlinear Evolution Equations and Painlevé Test, Singapore: World Scientific, 1988.

[15] W. Zhuang, Symbolic Computation of Exact Solutions of Nonlinear Evolution and Wave Equations. Master's Thesis, Dept. Math. and Comp. Sci., Colorado School of Mines, Golden, 1991.