MACSYMA PROGRAM FOR THE PAINLEVE TEST OF NONLINEAR ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

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1. INTRODUCTION*

A simple equation or system is said to have the Painlevé property, PP, if its only singularities in the complex plane consist of movable poles[1−4)]. This requires that the solution f of a PDE, say in two independent variables (t,x) can be expressed as

$$f = g^{\alpha} \sum_{k=0}^{\infty} u_k g^k, \tag{1}$$

with $u_0(t,x) \neq 0$, α a negative integer, and where $u_k(t,x)$ are analytic functions in a neighbourhood of the singular, non-characteristic manifold $g(t,x) = 0$, with $g_x(t,x) \neq 0$.

Performing the Painlevé test, as proposed by Weiss[3)], involves three steps:

(i) Determination of the negative integer α and u_0 from the leading order *ansatz* $f \propto u_0 g^{\alpha}$, by balancing the minimal power terms;

(ii) Indentification of the non-negative integer powers r, the *resonances*, at which arbitrary functions u_r can enter the expansion $f \propto u_0 g^{\alpha} + u_r g^{\alpha+r}$. This is achieved by requiring that u_r is arbitrary after substitution of this form for f into the equation, only retaining the most singular terms;

(iii) Verification that the correct number of arbitrary functions u_r indeed exists by substituting a truncated expansion of the form (1), $k = 1, 2, ..., rmax$, where $rmax$ represents the largest resonance, into the given equaton. At non-resonance

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* Sponsored by the Air Force Office of Scientific Research under Grant No. 85-NM-0263

levels, u_k should be unambiguously determined; at resonance levels u_r should be arbitrary due to a vanishing coefficient of $g^{r+minpowg}$ (compatibility condition). Here, $minpowg$ denotes the (negative) power in g of the most singular terms in the equation.

An equation or system for which the above steps can be carried out consistently, is said to have the PP and is conjectured to be integrable[1,5−7]. Counter examples[7−10] of integrable PDEs without the PP disprove the necessity of the PP for integrability (i. e. being exactly solvable by a linear integral equation of Inverse Scattering Transform); whereas some non - integrable equations seem to have the PP as defined by Weiss *et al*[3], hence questioning the conjecture[11] in its present form. Refinements of the PP have been established[12−15], allowing for rational power expansions of f, which lead to algebraic branch points in addition to movable poles.

Nevertheless, the PP has been a useful criterion for complete integrability tests[5,6,10−18]. It is a handy tool in the derivation of Bäcklund transformations and Lax pair representations for various PDEs[8.18−20], and it gives insight in Hirota's direct method for solitary wave construction[21,22], and other expansion methods which lead to rational solutions[18].

2. SCOPE AND LIMITATIONS OF THE PROGRAM

The program is largely based upon the MACSYMA routine *ODEPAINLEVE* developed by Rand and Winternitz[23], which checks the PP for a single ODE with real polynomial terms.

The package works for both single ODEs and PDEs, with arbitrary degree of nonlinearity. In principal, the number of variables is unlimited. Although the released version of the program works with only four independent variables. PDEs may have time and space dependent coefficients of integer degree. Transendental terms in the equation must be removed by a suitable exponential transformation of the dependent variable[17,24,25].

Due to 'intermediate expression swell' it may occur that the program can not perform the test in a reasonable time. Therefore, verifying the resonances and compatibility conditions are made optional through the boolian variable *do_resonances*. The highest level of verification is controllable by entering a value for *max_resonance*. For the investigation of Bäcklund transformations one may consider calculations beyond the automatically determined level *rmax* by taking $max\_resonance \geq rmax$.

Intermediate output can be obtained by including extra *print* statements in the program. (Warning : After any alternation in the main program a new LISP version needs to be created by *batch(main_p)* in MACSYMA).

Major difficulties arise when u_0 can not be substituted since it occurs in subsequent calculations at a lower degree than the one present at its evaluation. In that case, calculations by hand or in interactive MACSYMA are recommended. The vital steps for that are easily extracted from the program listing.

At present, the program does not perform the weakest forms of the Painlevé test[12−15] although α in (1) may be rational and can be positive, through a user supplied value for *beta*. Later extensions[26] will cover complex equations and coupled systems. Some of these must be treated by rather general rational power expansions[12−15]. A further refinement of the algorithm, performing selective substitutions of u_k and their derivatives in the recursive relations, will allow to construct Bäcklund transformations for PDEs[8,19,25].

3. HOW TO USE THE PROGRAM : EXAMPLES

The equation to be tested should be saved in a *batch* file, before entering MACSYMA.

Example 1 : An ODE, where the use of independent variable x is mandatory.

```
/* Batch file R_W_(5.6) for testing Eq. (5.6) in paper Rand-Winternitz23) */
    eq : (f^2)*fx[3](x) - 3*(fx[1](x))^3 $
    beta : -2 $
    max_resonance : 5 $
/* beta and max_resonance are specified here */
```

Note that, during the test the variable x will be replaced by $g = x - x_0$, where x_0 is the arbitrary initial value of x.

Example 2 : A PDE, with independent variables t, x, and y.

```
/* Batch file KP, for the Kadomtsev-Petviashvili Equation3) */
    eq : ftxy[1,1,0](t,x,y) + (ftxy[0,1,0](t,x,y))^2 + f*ftxy[0,2,0](t,x,y)
    + b*ftxy[0,4,0](t,x,y) + ftxy[0,0,2](t,x,y) $
/* b is a constant; beta, do_resonances and max_resonance are not specified */
```

Similarly, using $ftxyz[.,.,.,.](t,x,y,z)$, evolution and wave equations in t, x, y and z can be entered. For parametrical equations the explicit dependence on the variables must be given (e.g. coefficients $a(t), b(x), ...$). The *five first* letters of the alphabet are reserved to denote arbitrary constants, if necessary to be supplemented with $a1, a2, ..., b1, ...$

Once the batch files are made, enter MACSYMA and subsequently type :

(c1) batch(main_p) $ → to create *painleve.lsp*, only to be done ones
(c2) batch(setup_p) $ → to repeat before every new example
(c3) batch(kp) $ → reads in the batch file *kp*
(c5) writefile(test_kp) $ → opens output file *test_kp*
(c6) batch(exec_p) $ → starts the actual Painleve test
(c7) closefile() $ → closes output file *test_kp*

4. TEST RUNS

Although the program has been successfully tested, with eunice MACSYMA version 309.3 on a VAX 11/780, for more than 30 examples including all ODEs and selected PDEs from the cited references, unexpected situations may still occur. In case of trouble, the printout gives vital information to remedy the problems. The program does not rely on non-standard MACSYMA routines (e.g. *powers* is provided).

Model of Output :

For the Space Dependent Burgers equation[10,20] : $-x^2 f_{2x} - x f f_x + f_t = 0$, which passes the test :

The results of test runs on 15 other examples are summarised in Table 1.

5. LISTING

5.1 Listing of *setup_p* routine

5.2 Listing of *exec_p* routine

5.3 Listing of *main_p* routine

ACKNOWLEDGEMENTS

This work is partially supported by AFOSR under Grant 85-NM-0263. We thank Jonathan Len (Symbolics Inc.) for writing the POWERS subroutine.

6. REFERENCES

1) Ablowitz, M.J., Ramani, A. and Segur, H., *Lett. Nuovo Cim.* **23**, 333 (1978).
2) Ablowitz, M.J., Ramani, A. and Segur, H., *J. Math. Phys.* **21**, 715 (1980).
3) Weiss, J., Tabor, M. and Carnevale, G., *J. Math. Phys.* **24**, 522 (1983).
4) Ward, R.S., *Phys. Lett.* **102A**, 279 (1984).
5) McLeod, J.B. and Olver, P.J., *Siam J. Math. Anal.* **14**, 488 (1983).
6) Menyuk, C.R., Chen, H.H. and Lee, Y.C., *Phys. Rev. A* **27**, 1597 (1983).
7) Lakshmanan, M. and Kaliappan, P., *J. Math. Phys.* **24**, 795 (1983).
8) Weiss, J., *J. Math. Phys.* **24**, 1405 (1983).
9) Steeb, W.-H., Kloke, M., Spieker, B.M. and Grensing, D., *Prog. Theor. Phys.* **73**, 344 (1985).
10) Steeb, W.-H., and Louw, J.A., *Physica Scripta* **36**, 11 (1987).
11) Doktorov, E.V. and Sakovich, S. Yu., *J. Phys. A* **18**, 3327 (1985).
12) Dorizzi, B., Grammaticos, B. and Ramani, A., *J. Math. Phys.* **24**, 2282 (1983).
13) Grammaticos, B., Dorizzi, B. and Ramani, A., *J. Math. Phys.* **24**, 2289 (1983).
14) Graham, R., Roekaerts., D. and Tél, T., *Phys. Rev. A***31**, 3364 (1985).
15) Chowdhury, Roy A., Chanda, P.K. and Roy, S., *Prog. Theor. Phys.* **75**, 751 (1986).
16) Steeb, W.-H., Grauel, A., Kloke, M. and Spieker, B.M., *Physica Scripta* **31**, 5 (1985).
17) Steeb, W.-H., Kloke, M., Spieker, B.M. and Grensing, D., *Z. Phys. C* **28**, 241 (1985).
18) Weiss, J., *J. Math. Phys.* **25**, 13 (1984).
19) Weiss, J., *J. Math. Phys.* **27**, 1293 (1986).
20) Steeb, W.-H., Louw, J.A. and Strampp, W., *Prog. Theor. Phys. Lett.* **75**, 455 (1986).
21) Gibbon, J.D., Radmore, P., Tabor, M. and Wood, D., *Stud. Appl. Math.* **72**, 39 (1985).
22) Steeb, W.-H., Kloke, M. and Spieker, B.M., *J. Phys. A* **17**, L825 (1984).
23) Rand, D.W. and Winternitz, P., *Comp. Phys. Comm.* **42**, 359 (1986).
24) Steeb, W.-H., Kloke, M., Spieker, B.M. and Grauel, A., *Lett. Nuovo Cim.* **39**, 429 (1984).
25) Steeb, W.-H., Kloke, M. and Spieker, B.M., *Z. Naturforsch.* **38a**, 1054 (1983).
26) Hereman, W. and Van Den Bulck, E., *Comp. Phys. Comm.*, in preparation (1988).

27) Weiss, J., *J. Math. Phys.* **25**, 2226 (1984).
28) Hereman, W., Banerjee, P.P., Korpel, A., Assanto, G., Van Immerzeele, A. and Meerpoel, A., *J. Phys. A* **19**, 607 (1986).
29) Mahomed, F.M., *Ph.D. Thesis, University of the Witwatersrand, Johannesburg, South Africa*, **App. C**, 88 (1986).
30) Hlavatý, L., *Comp. Phys. Comm.* **42**, 427 (1986).