Poster Presentation

Symbolic Computation of Conserved Densities for Systems of Nonlinear Evolution and Differential-difference Equations

Ünal Göktaş and Willy Hereman

Mathematical and Computer Sciences
Colorado School of Mines
Golden, CO 80401-1887

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- **Purpose**

  Design and implement an algorithm to compute polynomial conservation laws for nonlinear systems of evolution and differential-difference equations

- **Motivation**

  – Conservation laws describe the conservation of fundamental physical quantities such as linear momentum and energy. Compare with constants of motion (first integrals) in mechanics

  – For nonlinear PDEs and DDEs, the existence of a sufficiently large (in principal infinite) number of conservation laws assures complete integrability

  – Conservation laws provide a simple and efficient method to study both quantitative and qualitative properties of equations and their solutions, e.g. Hamiltonian structures

  – Conservation laws can be used to test numerical integrators

PART I: PDEs

- **Conservation Laws for PDEs**

Consider a single nonlinear evolution equation

$$u_t = \mathcal{F}(u, u_x, u_{xx}, ..., u_{nx})$$

Conservation law:

$$\rho_t + J_x = 0$$

ρ is the density, J is the flux

Both are polynomial in u and its x derivatives

Consequently

$$P = \int_{-\infty}^{+\infty} \rho \, dx = \text{constant}$$

if J vanishes at infinity

- **Example**

Consider the Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{3x} = 0$$

Conserved densities

$$\rho_1 = u, \qquad (u)_t + (\frac{u^2}{2} + u_{2x})_x = 0$$

$$\rho_2 = u^2, \qquad (u^2)_t + (\frac{2u^3}{3} + 2uu_{2x} - {u_x}^2)_x = 0$$

$$\rho_3 = u^3 - 3{u_x}^2,$$

$$\left(u^3 - 3{u_x}^2\right)_t + \left(\frac{3}{4}u^4 - 6u{u_x}^2 + 3u^2 u_{2x} + 3{u_{2x}}^2 - 6u_x u_{3x}\right)_x = 0$$

$\vdots$

$$\rho_6 = u^6 - 60u^3{u_x}^2 - 30{u_x}^4 + 108u^2{u_{2x}}^2$$

$$+\frac{720}{7}{u_{2x}}^3 - \frac{648}{7}u{u_{3x}}^2 + \frac{216}{7}{u_{4x}}^2, \qquad \text{...... long ......}$$

$\vdots$

Note: KdV equation is invariant under the scaling symmetry

$$(x, t, u) \rightarrow (\lambda x, \lambda^3 t, \lambda^{-2} u)$$

u (and t) carry the weight of 2 (resp. 3) derivatives with respect to x

$$u \sim \frac{\partial^2}{\partial x^2}, \qquad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3}$$

- **Key Steps of the Algorithm**

  1. Determine weights (scaling properties) of variables & parameters

  2. Construct the form of the density (building blocks)

  3. Determine the unknown constant coefficients

- **Example:** For the KdV equation, compute the density of rank 6

  (i) Take all the variables, except $(\frac{\partial}{\partial t})$, with positive weight.
  Here, only u with $w(u) = 2$

  List all possible powers of u, up to rank 6 : $[u, u^2, u^3]$

  Introduce x derivatives to 'complete' the rank

  u has weight 2, introduce $\frac{\partial^4}{\partial x^4}$,

  u^2 has weight 4, introduce $\frac{\partial^2}{\partial x^2}$,

  u^3 has weight 6, no derivative needed

  (ii) Apply the derivatives
  Remove terms that are total derivatives with respect to x
  or total derivative up to terms kept earlier in the list

$$
\begin{aligned}
[u_{4x}] &\rightarrow [\,]\ \text{empty list} \\
[{u_x}^2, uu_{2x}] &\rightarrow [{u_x}^2]\ \ (uu_{2x} = (uu_x)_x - {u_x}^2) \\
[u^3] &\rightarrow [u^3]
\end{aligned}
$$

Combine the 'building blocks'

$$\rho = c_1 u^3 + c_2 {u_x}^2$$

(iii) Determine the coefficients c_1 and c_2

1. Compute $\dfrac{\partial \rho}{\partial t} = 3c_1 u^2 u_t + 2c_2 u_x u_{xt}$,
2. Replace u_t by $-(uu_x + u_{3x})$ and u_{xt} by $-(uu_x + u_{3x})_x$
3. Integrate the result with respect to x

   Carry out all integrations by parts (or use the Euler operator)

$$\frac{\partial \rho}{\partial t} = -[\frac{3}{4}c_1 u^4 - (3c_1 - c_2)uu_x^2 + 3c_1 u^2 u_{2x} - c_2 {u_{2x}}^2 + 2c_2 u_x u_{3x}]_x \\ -(3c_1 + c_2){u_x}^3$$

4. The non-integrable (last) term must vanish. Thus, $c_1 = -\frac{1}{3}c_2$. Set $c_2 = -3$, hence, $c_1 = 1$

   Result:

$$\rho = u^3 - 3{u_x}^2$$

   Expression [...] yields

$$J = \frac{3}{4}u^4 - 6u{u_x}^2 + 3u^2 u_{2x} + 3{u_{2x}}^2 - 6u_x u_{3x}$$

- **Application**

A Class of Fifth-Order Evolution Equations

$$u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0$$

where α, β, γ are nonzero parameters

$$u \sim \frac{\partial^2}{\partial x^2}$$

Special cases:

$\alpha = 30$	$\beta = 20$	$\gamma = 10$	Lax
$\alpha = 5$	$\beta = 5$	$\gamma = 5$	Sawada − Kotera
$\alpha = 20$	$\beta = 25$	$\gamma = 10$	Kaup−Kupershmidt
$\alpha = 2$	$\beta = 6$	$\gamma = 3$	Ito

Under what conditions for the parameters α, β and γ does this equation admit a density of fixed rank?

- **Rank 2:**
  No condition

$$\rho = u$$

- **Rank 4:**
  Condition: $\beta = 2\gamma$ (Lax and Ito cases)

$$\rho = u^2$$

– **Rank 6:**

Condition:

$$10\alpha = -2\beta^2 + 7\beta\gamma - 3\gamma^2$$

(Lax, SK, and KK cases)

$$\rho = u^3 + \frac{15}{(-2\beta + \gamma)} {u_x}^2$$

– **Rank 8:**

1. $\beta = 2\gamma$ (Lax and Ito cases)

$$\rho = u^4 - \frac{6\gamma}{\alpha} u {u_x}^2 + \frac{6}{\alpha} {u_{2x}}^2$$

2. $\alpha = -\frac{2\beta^2 - 7\beta\gamma - 4\gamma^2}{45}$ (SK, KK and Ito cases)

$$\rho = u^4 - \frac{135}{2\beta + \gamma} u {u_x}^2 + \frac{675}{(2\beta + \gamma)^2} {u_{2x}}^2$$

– **Rank 10:**

Condition:

$$\beta = 2\gamma$$

and

$$10\alpha = 3\gamma^2$$

(Lax case)

$$\rho = u^5 - \frac{50}{\gamma} u^2 {u_x}^2 + \frac{100}{\gamma^2} u {u_{2x}}^2 - \frac{500}{7\gamma^3} {u_{3x}}^2.$$

What are the necessary conditions for the parameters α, β and γ for this equation to admit ∞ many polynomial conservation laws?

- If $\alpha = \frac{3}{10}\gamma^2$ and $\beta = 2\gamma$ then there is a sequence (without gaps!) of conserved densities (Lax case)

- If $\alpha = \frac{1}{5}\gamma^2$ and $\beta = \gamma$ then there is a sequence (with gaps!) of conserved densities (SK case)

- If $\alpha = \frac{1}{5}\gamma^2$ and $\beta = \frac{5}{2}\gamma$ then there is a sequence (with gaps!) of conserved densities (KK case)

- If

  $$\alpha = -\frac{2\beta^2 - 7\beta\gamma + 4\gamma^2}{45}$$

  or

  $$\beta = 2\gamma$$

  then there is a conserved density of rank 8

  Combine both conditions: $\alpha = \frac{2\gamma^2}{9}$ and $\beta = 2\gamma$ (Ito case)

PART II: DDEs

- **Conservation Laws for DDEs**

Consider a system of DDEs, continuous in time, discretized in one space variable,

$$\dot{\mathbf{u}}_n = \mathbf{F}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$$

$\mathbf{u}_n$ and $\mathbf{F}$ are vector dynamical variables

Conservation law:

$$\dot{\rho}_n = J_n - J_{n+1}$$

ρ_n is the density and J_n is the flux

Both are polynomials in $\mathbf{u}_n$ and its shifts

$\frac{\mathrm{d}}{\mathrm{dt}}(\Sigma_n \, \rho_n) = \Sigma_n \, \dot{\rho}_n = \Sigma_n (J_n - J_{n+1})$, and if J_n is bounded for all n, with suitable boundary conditions,

$$\sum_n \rho_n = \text{constant}$$

- **Definitions**

D is the *shift-down* operator and U is the *shift-up* operator.
$Dm = m|_{n\to n-1}$ and $Um = m|_{n\to n+1}$.
For example, $Du_{n+2}v_n = u_{n+1}v_{n-1}$ and $Uu_{n-2}v_{n-1} = u_{n-1}v_n$.
Compositions of D and U define an *equivalence relation*
All shifted monomials are *equivalent*, e.g.

$$u_{n-1}v_{n+1} \equiv u_{n+2}v_{n+4} \equiv u_{n-3}v_{n-1}$$

Use the following *equivalence criterion*:
If two monomials, m_1 and m_2, are equivalent, $m_1 \equiv m_2$, then

$$m_1 = m_2 + [M_n - M_{n+1}]$$

for some polynomial M_n
For example, $u_{n-2}u_n \equiv u_{n-1}u_{n+1}$ since

$$u_{n-2}u_n = u_{n-1}u_{n+1} + [u_{n-2}u_n - u_{n-1}u_{n+1}] = u_{n-1}u_{n+1} + [M_n - M_{n+1}],$$

with $M_n = u_{n-2}u_n$.

Main representative of an equivalence class; the monomial with label n on u (or v)

For example, u_nu_{n+2} is the main representative of the class with elements $u_{n-1}u_{n+1}, u_{n+1}u_{n+3}$, etc.

Use lexicographical ordering to resolve conflicts

For example, u_nv_{n+2} (not $u_{n-2}v_n$) is the main representative in the class with elements $u_{n-3}v_{n-1}, u_{n+2}v_{n+4}$, etc.

- **Algorithm**

Consider the Toda lattice

$$\dot{u}_n = v_{n-1} - v_n, \qquad \dot{v}_n = v_n(u_n - u_{n+1})$$

Can compute a couple of conservation laws by hand:

$$\dot{u}_n = \dot{\rho}_n = v_{n-1} - v_n = J_n - J_{n+1}$$

with $J_n = v_{n-1}$.

Denote this *first* pair by

$$\rho_n^{(1)} = u_n, \qquad J_n^{(1)} = v_{n-1}$$

A *second* pair:

$$\rho_n^{(2)} = \frac{1}{2}{u_n}^2 + v_n, \qquad J_n^{(2)} = u_n v_{n-1}$$

Key observation: DDE and the above pairs, are invariant under the scaling symmetry

$$(t, u_n, v_n) \to (\lambda t, \lambda^{-1} u_n, \lambda^{-2} v_n)$$

Result of this dimensional analysis: u_n corresponds to one derivative with respect to t

For short, $u_n \sim \frac{\mathrm{d}}{\mathrm{dt}}$, and similarly, $v_n \sim \frac{\mathrm{d}^2}{\mathrm{dt}^2}$

Our algorithm exploits this symmetry to find conserved densities, which has three steps:

1. Determining the weights,
2. Constructing the form of density,
3. Determining the unknown coefficients.

- **Step 1: Determine the weights**

  The *weight*, w, of a variable is equal to the number of derivatives with respect to t the variable carries

  Weights are positive, rational, and independent of n

  Set $w(\frac{\mathrm{d}}{\mathrm{dt}}) = 1$

  For the Toda lattice, $w(u_n) = 1$, and $w(v_n) = 2$

  The *rank* of a monomial is the total weight of the monomial, in terms of derivatives with respect to t

  In each equation of the Toda lattice, all the terms are *uniform* in rank

  Requiring uniformity in rank for each equation, allows one to compute the weights of the dependent variables

  Indeed,

  $$w(u_n) + 1 = w(v_n), \quad w(v_n) + 1 = w(u_n) + w(v_n),$$

  yields

  $$w(u_n) = 1, \quad w(v_n) = 2,$$

  consistent with the scaling symmetry

- **Step 2: Construct the form of the density**

  For example, compute the form of the density of rank 3

  List all monomials in u_n and v_n of rank 3 or less:

$$\mathcal{G} = \{{u_n}^3, {u_n}^2, u_n v_n, u_n, v_n\}$$

  Next, for each monomial in $\mathcal{G}$, introduce enough t-derivatives, so that each term exactly has weight 3. Using the equations of Toda lattice,

$$\frac{\mathrm{d}^0}{\mathrm{dt}^0}({u_n}^3) = {u_n}^3, \qquad \frac{\mathrm{d}^0}{\mathrm{dt}^0}(u_n v_n) = u_n v_n,$$
$$\frac{\mathrm{d}}{\mathrm{dt}}({u_n}^2) = 2u_n v_{n-1} - 2u_n v_n, \qquad \frac{\mathrm{d}}{\mathrm{dt}}(v_n) = u_n v_n - u_{n+1} v_n,$$
$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}(u_n) = u_{n-1} v_{n-1} - u_n v_{n-1} - u_n v_n + u_{n+1} v_n$$

  Gather the resulting terms in a set

$$\mathcal{H} = \{{u_n}^3, u_n v_{n-1}, u_n v_n, u_{n-1} v_{n-1}, u_{n+1} v_n\}$$

  Identify members that belong to the same equivalence classes and replace them by the main representatives.

  For example, since $u_n v_{n-1} \equiv u_{n+1} v_n$ both are replaced by $u_n v_{n-1}$.
  $\mathcal{H}$ is replaced by

$$\mathcal{I} = \{{u_n}^3, u_n v_{n-1}, u_n v_n\},$$

  containing the building blocks of the density.

  Linear combination of the monomials in $\mathcal{I}$ with constant coefficients c_i gives

$$\rho_n = c_1\, {u_n}^3 + c_2\, u_n v_{n-1} + c_3\, u_n v_n$$

- **Step 3: Determine the unknown coefficients**

Require that conservation law holds

Compute $\dot{\rho}_n$.

Use the equations to remove $\dot{u}_n, \dot{v}_n$, etc.

Group the terms

$$\begin{aligned}\dot{\rho}_n &= (3c_1 - c_2){u_n}^2 v_{n-1} + (c_3 - 3c_1){u_n}^2 v_n + (c_3 - c_2) v_{n-1} v_n \\ &\quad + c_2 u_{n-1} u_n v_{n-1} + c_2 {v_{n-1}}^2 - c_3 u_n u_{n+1} v_n - c_3 {v_n}^2\end{aligned}$$

Use the equivalence criterion to modify $\dot{\rho}_n$. Replace $u_{n-1} u_n v_{n-1}$ by $u_n u_{n+1} v_n + [u_{n-1} u_n v_{n-1} - u_n u_{n+1} v_n]$. The goal is to introduce the main representatives. Therefore,

$$\begin{aligned}\dot{\rho}_n &= (3c_1 - c_2){u_n}^2 v_{n-1} + (c_3 - 3c_1){u_n}^2 v_n \\ &\quad + (c_3 - c_2) v_n v_{n+1} + [(c_3 - c_2) v_{n-1} v_n - (c_3 - c_2) v_n v_{n+1}] \\ &\quad + c_2 u_n u_{n+1} v_n + [c_2 u_{n-1} u_n v_{n-1} - c_2 u_n u_{n+1} v_n] \\ &\quad + c_2 {v_n}^2 + [c_2 {v_{n-1}}^2 - c_2 {v_n}^2] - c_3 u_n u_{n+1} v_n - c_3 {v_n}^2\end{aligned}$$

Group the terms outside of the square brackets and move the pairs inside the square brackets to the bottom. Rearrange the latter terms so that they match the pattern $[J_n - J_{n+1}]$. Hence,

$$\begin{aligned}\dot{\rho}_n &= (3c_1 - c_2){u_n}^2 v_{n-1} + (c_3 - 3c_1){u_n}^2 v_n \\ &\quad + (c_3 - c_2) v_n v_{n+1} + (c_2 - c_3) u_n u_{n+1} v_n + (c_2 - c_3){v_n}^2 \\ &\quad + [\{(c_3 - c_2) v_{n-1} v_n + c_2 u_{n-1} u_n v_{n-1} + c_2 {v_{n-1}}^2\} \\ &\quad - \{(c_3 - c_2) v_n v_{n+1} + c_2 u_n u_{n+1} v_n + c_2 {v_n}^2\}]\end{aligned}$$

The terms inside the square brackets determine:

$$J_n = (c_3 - c_2)v_{n-1}v_n + c_2 u_{n-1}u_n v_{n-1} + c_2 {v_{n-1}}^2$$

The terms outside the square brackets must all vanish, yielding

$$\mathcal{S} = \{3c_1 - c_2 = 0, c_3 - 3c_1 = 0, c_2 - c_3 = 0\}$$

The solution is $3c_1 = c_2 = c_3$. Choose $c_1 = \frac{1}{3}$, $c_2 = c_3 = 1$,

$$\rho_n = \tfrac{1}{3}\, {u_n}^3 + u_n(v_{n-1} + v_n), \qquad J_n = u_{n-1}u_n v_{n-1} + {v_{n-1}}^2$$

Analogously, conserved densities of rank ≤ 5:

$$\rho_n^{(1)} = u_n, \qquad \rho_n^{(2)} = \tfrac{1}{2}{u_n}^2 + v_n,$$

$$\rho_n^{(3)} = \tfrac{1}{3}{u_n}^3 + u_n(v_{n-1} + v_n),$$

$$\rho_n^{(4)} = \tfrac{1}{4}{u_n}^4 + {u_n}^2(v_{n-1} + v_n) + u_n u_{n+1} v_n + \tfrac{1}{2}{v_n}^2 + v_n v_{n+1},$$

$$\begin{aligned}\rho_n^{(5)} = {} & \tfrac{1}{5}{u_n}^5 + {u_n}^3(v_{n-1} + v_n) + u_n u_{n+1} v_n(u_n + u_{n+1}) \\ & + u_n v_{n-1}(v_{n-2} + v_{n-1} + v_n) + u_n v_n(v_{n-1} + v_n + v_{n+1})\end{aligned}$$

- **Application**

  Parameterized Toda lattice:

  $$\dot{u}_n = \alpha\, v_{n-1} - v_n, \quad \dot{v}_n = v_n\,(\beta\, u_n - u_{n+1}),$$

  α and β are *nonzero* parameters, and integrable if $\alpha = \beta = 1$

  Compute the *compatibility conditions* for α and β, so that there is a conserved densities of, say, rank 3.

  In this case, we have $\mathcal{S}$:

  $\{3\alpha c_1 - c_2 = 0, \beta c_3 - 3c_1 = 0, \alpha c_3 - c_2 = 0, \beta c_2 - c_3 = 0, \alpha c_2 - c_3 = 0\}$

  A non-trivial solution $3c_1 = c_2 = c_3$ will exist *iff* $\alpha = \beta = 1$

  Analogously, parameterized Toda lattice has density

  $\rho_n^{(1)} = u_n$ of rank 1 if $\alpha = 1$,

  and density

  $\rho_n^{(2)} = \frac{\beta}{2} {u_n}^2 + v_n$ of rank 2 if $\alpha\,\beta = 1$

  Only when $\alpha = \beta = 1$ will the parameterized system have conserved densities of rank ≥ 3

- **Example: Nonlinear Schrödinger (NLS) equation**

  Ablowitz and Ladik discretization of the NLS equation:

$$i\,\dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + u_n^* u_n (u_{n+1} + u_{n-1}),$$

  where u_n^* is the complex conjugate of u_n. Treat u_n and $v_n = u_n^*$ as independent variables and add the complex conjugate equation. Absorbing i in the scale on t, gives

$$\begin{aligned}\dot{u}_n &= u_{n+1} - 2u_n + u_{n-1} + u_n v_n (u_{n+1} + u_{n-1}),\\ \dot{v}_n &= -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1})\end{aligned}$$

  Since $v_n = u_n^*$, $w(v_n) = w(u_n)$.

  No uniformity in rank! Circumvent this problem by introducing an auxiliary parameter α with weight,

$$\begin{aligned}\dot{u}_n &= \alpha(u_{n+1} - 2u_n + u_{n-1}) + u_n v_n (u_{n+1} + u_{n-1}),\\ \dot{v}_n &= -\alpha(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}).\end{aligned}$$

  Uniformity in rank requires that

$$\begin{aligned}w(u_n) + 1 &= w(\alpha) + w(u_n) = 2w(u_n) + w(v_n) = 3w(u_n),\\ w(v_n) + 1 &= w(\alpha) + w(v_n) = 2w(v_n) + w(u_n) = 3w(v_n),\end{aligned}$$

  which yields

$$w(u_n) = w(v_n) = \frac{1}{2}, w(\alpha) = 1.$$

Uniformity in rank is essential for the first two steps of the algorithm. After Step 2, set $\alpha = 1$

The computations now proceed as in the previous examples

Conserved densities:

$$
\begin{aligned}
\rho_n^{(1)} &= c_1 u_n v_{n-1} + c_2 u_n v_{n+1}, \\
\rho_n^{(2)} &= c_1 \left(\tfrac{1}{2} {u_n}^2 {v_{n-1}}^2 + u_n u_{n+1} v_{n-1} v_n + u_n v_{n-2}\right) \\
&+ c_2 \left(\tfrac{1}{2} {u_n}^2 {v_{n+1}}^2 + u_n u_{n+1} v_{n+1} v_{n+2} + u_n v_{n+2}\right), \\
\rho_n^{(3)} &= c_1 \big[\tfrac{1}{3} {u_n}^3 {v_{n-1}}^3 \\
&\quad + u_n u_{n+1} v_{n-1} v_n \left(u_n v_{n-1} + u_{n+1} v_n + u_{n+2} v_{n+1}\right) \\
&\quad + u_n v_{n-1} \left(u_n v_{n-2} + u_{n+1} v_{n-1}\right) \\
&\quad + u_n v_n \left(u_{n+1} v_{n-2} + u_{n+2} v_{n-1}\right) + u_n v_{n-3}\big] \\
&+ c_2 \big[\tfrac{1}{3} {u_n}^3 {v_{n+1}}^3 \\
&\quad + u_n u_{n+1} v_{n+1} v_{n+2} \left(u_n v_{n+1} + u_{n+1} v_{n+2} + u_{n+2} v_{n+3}\right) \\
&\quad + u_n v_{n+2} \left(u_n v_{n+1} + u_{n+1} v_{n+2}\right) \\
&\quad + u_n v_{n+3} \left(u_{n+1} v_{n+1} + u_{n+2} v_{n+2}\right) + u_n v_{n+3}\big].
\end{aligned}
$$

- **Scope and Limitations**

  - Systems must be polynomial in dependent variables
  - Only two independent variables (x and t) are allowed
  - No terms should *explicitly* depend on x and t
  - Program only computes polynomial-type conserved densities; only polynomials in the dependent variables and their derivatives; no explicit dependencies on x and t
  - No limit on the number of evolution equations and DDEs. In practice: time and memory constraints
  - Input systems may have (nonzero) parameters. Program computes the conditions for parameters such that densities (of a given rank) exist
  - Systems can also have parameters with (unknown) weight. Allows one to test systems with non-uniform rank
  - For systems where one or more of the weights are free. Program prompts the user to enter values for the free weights
  - Negative weights are not allowed
  - Fractional weights are permitted
  - Form of ρ can be given (testing purposes)

- **Conclusions and Further Research**
  - *Mathematica* programs *condens.m* and *diffdens.m*
  - Analysis of class of parameterized equations
  - Indicator of integrability
  - Exploit other symmetries in the hope to find conserved densities of non-polynomial form

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- In collaboration with Grant Erdmann

- Papers submitted to: J. Symb. Comp. and Phys. Lett. A

- Software: via ftp site *mines.edu*
  in subdirectory pub/papers/math_cs_dept/software/condens
  or via Internet URL: http://www.mines.edu/fs_home/whereman/