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IMACS PDE7 — SECOND TALK

REVIEW OF SYMBOLIC SOFTWARE FOR CALCULATING LIE SYMMETRIES OF PARTIAL DIFFERENTIAL EQUATIONS

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Calculation of Lie-point and Generalized Symmetries

- SPDE by Schwarz (Reduce, Scratchpad, 1986)
- Symmetries via exterior differential forms by Kersten and Gragert (Reduce, 1987)
- Lie-Bäcklund symmetries by Fedorova, Kornyak and Fushchich (Reduce, 1987)
- Crackstar by Wolf (Formac, 1987)
- Lie-point symmetries by Schwarzmeier and Rosenau (Macsyma, 1988)
- Special symmetries by Mikhailov (Pascal, 1988)

- LIE by Head (muMath, 1990)
- Lie program by Nucci (Reduce, 1990)
- PDELIE by Vafeades (Macsyma, 1990)
- DEliA by Bocharov (Pascal, 1990)
- SYM_DE by Steinberg (Macsyma, 1990)
- SYMCAL by Reid (Macsyma, 1990)
- SYMMGRP.MAX by Champagne, Hereman and Winternitz (Macsyma, 1990)
- Lie symmetries by Herod, Berube, Wilcox (Mathematica, in development 1992)

Example 1: The Korteweg-de Vries Equation

$$u_t + auu_x + u_{xxx} = 0$$

- one equation (parameter a)
- two independent variables t and x
- one dependent variable u
- vector field $\alpha = \eta^x \frac{\partial}{\partial x} + \eta^t \frac{\partial}{\partial t} + \varphi^u \frac{\partial}{\partial u}$

Format for SYMMGRP.MAX

- variables $x[1] = x, \quad x[2] = t, \quad u[1] = u$
- equation $e1 : u[1,[0,1]] + a * u[1] * u[1,[1,0]] + u[1,[3,0]]$
- variable to be eliminated $v1 : u[1,[0,1]]$
- coefficients of vectorfield in SYMMGRP.MAX: eta[1] $= \eta^x$, eta[2] $= \eta^t$ and phi[1] $= \varphi^u$

Format for PDELIE and SYM_DE

- variables: $[x, t]$ and $[u]$
- equation: $'\text{diff}(u, t) + a * u * '\text{diff}(u, x) + '\text{diff}(u, x, 3) = 0$
- variable to be eliminated: $'\text{diff}(u, t)$
- coefficients of vectorfield in PDELIE:
  $\text{GX} = \eta^x, \quad \text{GT} = \eta^t \quad$ and $\text{FU} = \varphi^u$
- coefficients of vectorfield in SYM_DE:
  $\text{X}_1 = \eta^x, \quad \text{X}_2 = \eta^t \quad$ and $\text{Y}_1 = \varphi^u$

Format for SPDE and LIE

- variables $X1 = x, \quad X2 = t, \quad U(1) = u$
- equation $e1 : U(1, 2) + A * U(1) * U(1, 1) + U(1, 1, 1, 1)$
- variable to be eliminated $U(1, 2)$
- coefficients of vectorfield in SPDE:
  $\text{XI}(1) = \eta^x, \quad \text{XI}(2) = \eta^t \quad$ and $\text{ETA}(1) = \varphi^u$
- coefficients of vectorfield in LIE:
  $\text{XXX\#1} = \eta^x, \quad \text{XXX\#2} = \eta^t \quad$ and $\text{UUU\#1} = \varphi^u$

There are only eight determining equations

$$\frac{\partial \mathrm{eta}[2]}{\partial u[1]} = 0$$

$$\frac{\partial \mathrm{eta}[2]}{\partial x[1]} = 0$$

$$\frac{\partial \mathrm{eta}[1]}{\partial u[1]} = 0$$

$$\frac{\partial^2 \mathrm{phi}[1]}{\partial u[1]^2} = 0$$

$$\frac{\partial^2 \mathrm{phi}[1]}{\partial u[1] \partial x[1]} - \frac{\partial^2 \mathrm{eta}[1]}{\partial x[1]^2} = 0$$

$$\frac{\partial \mathrm{phi}[1]}{\partial x[2]} + \frac{\partial^3 \mathrm{phi}[1]}{\partial x[1]^3} + u[1] \frac{\partial \mathrm{phi}[1]}{\partial x[1]} = 0$$

$$3\frac{\partial^3 \text{phi}[1]}{\partial u[1]\partial x[1]^2} - \frac{\partial \text{eta}[1]}{\partial x[2]} - \frac{\partial^3 \text{eta}[1]}{\partial x[1]^3} + 2\, u[1]\, \frac{\text{eta}[1]}{\partial x[1]} + \text{phi}[1] = 0$$

$$u[1]\, \frac{\partial \text{eta}[2]}{\partial x[2]} + 3\frac{\partial^3 \text{phi}[1]}{\partial u[1]\partial x[1]^2} - \frac{\partial \text{eta}[1]}{\partial x[2]} - \frac{\partial^3 \text{eta}[1]}{\partial x[1]^3} - u[1]\, \frac{\text{eta}[1]}{\partial x[1]} + \text{phi}[1] = 0$$

The solution in the original variables

$$\begin{aligned} \eta^x &= k_1 + k_3\, t - k_4\, x \\ \eta^t &= k_2 - 3k_4\, t \\ \varphi^u &= k_3 + 2\, k_4\, u \end{aligned}$$

The four infinitesimal generators are

$$\begin{aligned} G_1 &= \partial_x \\ G_2 &= \partial_t \\ G_3 &= t\, \partial_x + \partial_u \\ G_4 &= x\, \partial_x + 3\, t\, \partial_t - 2\, u\, \partial_u \end{aligned}$$

Equation is invariant under:

- translations G_1 and G_2
- Galilean boost G_3
- scaling G_4

Computation of the flows corresponding to G_1 thru G_4 shows that for any solution $u = f(x,t)$ of the KdV equation the transformed solutions

$$\tilde{u} = f(x-\epsilon, t)$$

$$\tilde{u} = f(x, t-\epsilon)$$

$$\tilde{u} = f(x-\epsilon, t) + \epsilon$$

$$\tilde{u} = e^{-2\epsilon} f(e^{-\epsilon}x, e^{-3\epsilon}t)$$

will solve the KdV equation

Note that ϵ is the parameter of the transformation group

Example 2: The Harry Dym Equation

$$u_t - u^3 u_{xxx} = 0$$

- one equation
- two independent variables t and x
- one dependent variable u
- vector field $\alpha = \eta^x \frac{\partial}{\partial x} + \eta^t \frac{\partial}{\partial t} + \varphi^u \frac{\partial}{\partial u}$

Format for SYMMGRP.MAX

- variables $x[1] = x, \quad x[2] = t, \quad u[1] = u$
- equation $e1 : u[1,[0,1]] - u[1]^3 * u[1,[3,0]]$
- variable to be eliminated $v1 : u[1,[0,1]]$
- coefficients of vectorfield in SYMMGRP.MAX:
  eta[1] $= \eta^x$, eta[2] $= \eta^t$ and phi[1] $= \varphi^u$

Format for PDELIE and SYM_DE

- variables: $[x, t]$ and $[u]$
- equation: $'\text{diff}(u, t) - u^3 * '\text{diff}(u, x, 3) = 0$
- variable to be eliminated: $'\text{diff}(u, t)$
- coefficients of vectorfield in PDELIE:
  $\text{GX} = \eta^x, \quad \text{GT} = \eta^t$ and $\text{FU} = \varphi^u$
- coefficients of vectorfield in SYM_DE:
  $\text{X}_1 = \eta^x, \quad \text{X}_2 = \eta^t$ and $\text{Y}_1 = \varphi^u$

Format for SPDE and LIE

- variables $X1 = x, \quad X2 = t, \quad U(1) = u$
- equation $e1 : U(1, 2) - U(1)^3 * U(1, 1, 1, 1)$
- variable to be eliminated $U(1, 2)$
- coefficients of vectorfield in SPDE:
  $\text{XI}(1) = \eta^x, \quad \text{XI}(2) = \eta^t$ and $\text{ETA}(1) = \varphi^u$
- coefficients of vectorfield in LIE:
  $\text{XXX\#1} = \eta^x, \quad \text{XXX\#2} = \eta^t$ and $\text{UUU\#1} = \varphi^u$

There are only eight determining equations,
in SYMMGRP.MAX notation

$$\frac{\partial \mathrm{eta}[2]}{\partial u[1]} = 0$$

$$\frac{\partial \mathrm{eta}[2]}{\partial x[1]} = 0$$

$$\frac{\partial \mathrm{eta}[1]}{\partial u[1]} = 0$$

$$\frac{\partial^2 \mathrm{phi}[1]}{\partial u[1]^2} = 0$$

$$\frac{\partial^2 \mathrm{phi}[1]}{\partial u[1] \partial x[1]} - \frac{\partial^2 \mathrm{eta}[1]}{\partial x[1]^2} = 0$$

$$\frac{\partial \mathrm{phi}[1]}{\partial x[2]} - u[1]^3 \frac{\partial^3 \mathrm{phi}[1]}{\partial x[1]^3} = 0$$

$$3u[1]^3 \frac{\partial^3 \mathrm{phi}[1]}{\partial u[1] \partial x[1]^2} + \frac{\partial \mathrm{eta}[1]}{\partial x[2]} - u[1]^3 \frac{\partial^3 \mathrm{eta}[1]}{\partial x[1]^3} = 0$$

$$u[1] \frac{\partial \mathrm{eta}[2]}{\partial x[2]} - 3u[1] \frac{\partial \mathrm{eta}[1]}{\partial x[1]} + 3 \,\mathrm{phi}[1] = 0$$

The solution in the original variables

$$\begin{aligned}
\eta^x &= k_1 + k_3\, x + k_5\, x^2 \\
\eta^t &= k_2 - 3k_4\, t \\
\varphi^u &= (k_3 + k_4 + 2k_5\, x)\, u
\end{aligned}$$

The five infinitesimal generators are

$$\begin{aligned}
G_1 &= \partial_x \\
G_2 &= \partial_t \\
G_3 &= x\partial_x + u\partial_u \\
G_4 &= -3t\partial_t + u\partial_u \\
G_5 &= x^2\partial_x + 2xu\partial_u
\end{aligned}$$

Equation is invariant under:

- translations G_1 and G_2
- scaling G_3 and G_4
- what else ?

Computation of the flow corresponding to G_5

$$\begin{aligned} \frac{d\tilde{x}}{d\epsilon} &= \tilde{x}^2 & \tilde{x}(0) &= x \\ \frac{d\tilde{t}}{d\epsilon} &= 0 & \tilde{t}(0) &= t \\ \frac{d\tilde{u}}{d\epsilon} &= 2\tilde{x}\tilde{u} & \tilde{u}(0) &= u \end{aligned} \tag{1}$$

ϵ is the parameter of the transformation group

One obtains

$$\begin{aligned} \tilde{x}(\epsilon) &= \frac{x}{1-\epsilon x} \\ \tilde{t}(\epsilon) &= t \\ \tilde{u}(\epsilon) &= \frac{u}{(1-\epsilon x)^2} \end{aligned}$$

Conclusion: for any solution $u = f(x,t)$ of the Harry Dym equation the transformed solution

$$\tilde{u}(\tilde{x},\tilde{t}) = (1+\epsilon\tilde{x})^2 f(\frac{\tilde{x}}{1+\epsilon\tilde{x}},\tilde{t})$$

will solve

$$\tilde{u}_{\tilde{t}} - \tilde{u}^3\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0$$

Example 3: The cylindrical KdV Equation

$$u_t + 6uu_x + u_{xxx} + \frac{1}{2t}u = 0$$

- one equation with time-dependent coefficient
- two independent variables t and x
- one dependent variable u
- vector field $\alpha = \eta^x \frac{\partial}{\partial x} + \eta^t \frac{\partial}{\partial t} + \varphi^u \frac{\partial}{\partial u}$

Format for SYMMGRP.MAX

- variables $x[1] = x, \quad x[2] = t, \quad u[1] = u$
- equation $e1 : u[1,[0,1]] + 6 * u[1] * u[1,[1,0]]$ $+u[1,[3,0]] + u[1]/(2 * x[2])$
- variable to be eliminated $v1 : u[1,[0,1]]$
- coefficients of vectorfield in SYMMGRP.MAX: eta[1] $= \eta^x$, eta[2] $= \eta^t$ and phi[1] $= \varphi^u$

There are only eight determining equations (not shown)

The solution in the original variables

$$
\begin{aligned}
\eta^x &= k_1 + k_2\, x + 2\, k_3\, \sqrt{t} + 2\, k_4\, x\, \sqrt{t} \\
\eta^t &= 3\, k_2\, t + 4\, k_4\, t\, \sqrt{t} \\
\varphi^u &= -2\, k_2\, u - 4\, k_4\, u\, \sqrt{t} + \frac{1}{6\sqrt{t}}\, (k_4\, x + k_3)
\end{aligned}
$$

The four infinitesimal generators are

$$
\begin{aligned}
G_1 &= \partial_x \\
G_2 &= x\, \partial_x + 3\, t\, \partial_t - 2\, u\, \partial_u \\
G_3 &= 12\, \sqrt{t}\, \partial_x + \frac{1}{\sqrt{t}}\, \partial_u \\
G_4 &= 12\, x\, \sqrt{t}\, \partial_x + 24\, t\, \sqrt{t}\, \partial_t - 24\sqrt{t}\, u\, \partial_u + x\, \frac{1}{\sqrt{t}}\, \partial_u
\end{aligned}
$$

Example 4: The Nonlinear Schrödinger Equation

$$iu_t + u_{xx} + 2|u|^2 u = 0$$

Split in real and imaginary parts via $u = v + iw$

$$\begin{aligned} v_t + v_{xx} + 2(v^2 + w^2)w &= 0 \\ w_t - w_{xx} - 2(v^2 + w^2)v &= 0 \end{aligned}$$

- two coupled equations
- two independent variables t and x
- two dependent variable v and w
- vector field $\alpha = \eta^x \frac{\partial}{\partial x} + \eta^t \frac{\partial}{\partial t} + \varphi^v \frac{\partial}{\partial v} + \varphi^w \frac{\partial}{\partial w}$

Format for SYMMGRP.MAX

- variables $x[1] = x, \quad x[2] = t, \quad u[1] = u, \quad u[2] = w$
- equations
  $e1 : u[1,[0,1]] + u[2,[2,0]] + 2 * u[2] * (u[1]^2 + u[2]^2)$
  $e2 : u[2,[0,1]] - u[1,[2,0]] - 2 * u[1] * (u[1]^2 + u[2]^2)$
- variables to be eliminated $v1 : u[1,[0,1]]$ and $v2 : u[2,[0,1]]$
- coefficients of vectorfield in SYMMGRP.MAX:
  eta[1] $= \eta^x$, eta[2] $= \eta^t$, phi[1] $= \varphi^v$ and phi[2] $= \varphi^w$

There are 20 determining equations

The solution in the original variables

$$\begin{aligned}
\eta^x &= k_1 + k_4\, x + 2\, k_5\, t \\
\eta^t &= k_2 + 2k_4\, t \\
\varphi^v &= -k_3\, w - k_4\, v - k_5\, x\, w \\
\varphi^w &= k_3\, v - k_4\, w + k_5\, x\, v
\end{aligned}$$

The five infinitesimal generators are

$$\begin{aligned}
G_1 &= \partial_x \\
G_2 &= \partial_t \\
G_3 &= v\, \partial_w - w\, \partial_v \\
G_4 &= 2\, t\, \partial_t + x\, \partial_x - v\, \partial_v - w\, \partial_w \\
G_5 &= 2\, t\, \partial_x - x\, w\, \partial_v + x\, v\, \partial_w
\end{aligned}$$

Equation is invariant under:

- translations G_1 and G_2
- rotation G_3
- dilation or scaling G_4
- Galilean boost or inversion G_5

Example 5: The Ernst Equations

$$
\begin{aligned}
F\nabla^2 F &= (\nabla F)^2 - (\nabla G)^2 \\
F\nabla^2 G &= 2(\nabla F)\cdot(\nabla G)^2
\end{aligned}
$$

With $F(\rho, z), G(\rho, z)$ and $\rho = \sqrt{x^2 + y^2}$

$$
\begin{aligned}
\nabla^2 F &= F_{\rho\rho} + \rho^{-1} + F_{zz} \\
|\nabla F|^2 &= F_\rho^2 + F_z^2
\end{aligned}
$$

$$
\begin{aligned}
F[\rho(F_{\rho\rho} + F_{zz}) + F_\rho] - \rho(F_\rho^2 + F_z^2 - G_\rho^2 - G_z^2) &= 0 \\
F[\rho(G_{\rho\rho} + G_{zz}) + G_\rho] - 2\rho(F_\rho G_\rho + F_z G_z) &= 0
\end{aligned}
$$

- two coupled equations
- two independent variables ρ and z
- two dependent variable F and G
- vector field $\alpha = \eta^\rho \frac{\partial}{\partial\rho} + \eta^z \frac{\partial}{\partial z} + \varphi^F \frac{\partial}{\partial F} + \varphi^G \frac{\partial}{\partial G}$

Format for SYMMGRP.MAX

- variables $x[1] = \rho, \quad x[2] = z, \quad u[1] = F, \quad u[2] = G$
- equations
  $e1 : u[1] * (x[1] * (u[1,[2,0]] + u[1,[0,2]]) + u[1,[1,0]])$
  $-x[1] * (u[1,[1,0]]^2 + u[1,[0,1]]^2 - u[2,[1,0]]^2 - u[2,[0,1]]^2)$

  $e2 : u[1] * (x[1] * (u[2,[2,0]] + u[2,[0,2]]) + u[2,[1,0]])$
  $-2 * x[1] * (u[1,[1,0]] * u[2,[1,0]] + u[1,[0,1]] * u[2,[0,1]])$

- variables to be eliminated $v1 : u[1,[2,0]]$ and $v2 : u[2,[2,0]]$
- coefficients of vectorfield in SYMMGRP.MAX:
  eta[1] $= \eta^\rho$, eta[2] $= \eta^z$, phi[1] $= \varphi^F$ and phi[2] $= \varphi^G$

There are 25 determining equations (not shown)

The solution in the original variables

$$\begin{aligned} \eta^\rho &= k_3\ \rho \\ \eta^z &= k_1 + k_3\ z \\ \varphi^F &= k_4\ F + 2\ k_5\ F\ G \\ \varphi^G &= k_2 + k_4\ G - k_5\ (F^2 - G^2) \end{aligned}$$

The five infinitesimal generators are

$$\begin{aligned} G_1 &= \partial_z \\ G_2 &= \partial_G \\ G_3 &= \rho\ \partial_\rho + z\ \partial_z \\ G_4 &= F\ \partial_F + G\ \partial_G \\ G_5 &= 2\ F\ G\ \partial_F - (F^2 - G^2)\ \partial_G \end{aligned}$$

Example 6: The Karpman Equations

The Karpman equations describe the effect of modulation instability of a high frequency (whistler) wave due to its resonant parametric interaction with a low frequency wave in a plasma

$$
\begin{aligned}
i(\psi_t + w_1\,\psi_z) + \frac{1}{2}[s_1\,(\psi_{xx} + \psi_{yy}) + s_2\,\psi_{zz}] - a_1\,\nu\,\psi &= 0 \\
\nu_{tt} - (w_2)^2\,(\nu_{xx} + \nu_{yy} + \nu_{zz}) - a_2\,(|\psi|^2_{xx} + |\psi|^2_{yy} + |\psi|^2_{zz}) &= 0
\end{aligned}
$$

ψ is the complex amplitude of the whistler wave
ν is the particle density of the plasma
a_1, a_2, s_1, s_2, w_1 and w_2 are real constants

Split ψ in real and imaginary parts

$$\psi = \rho(x,y,z,t)\exp[i\phi(x,y,z,t)]$$

$$\rho_t + w_1\rho_z + \left[\frac{s_1}{2}(2\rho_x\phi_x + 2\rho_y\phi_y + \rho\phi_{xx} + \rho\phi_{yy}) + \frac{s_2}{2}(2\rho_z\phi_z + \rho\phi_{zz})\right] = 0$$

$$\phi_t + w_1\phi_z - \left[\frac{s_1}{2}(\frac{\rho_{xx}}{\rho} + \frac{\rho_{yy}}{\rho} - {\phi_x}^2 - {\phi_y}^2) + \frac{s_2}{2}(\frac{\rho_{zz}}{\rho} - {\phi_z}^2)\right] + a_1\nu = 0$$

$$\nu_{tt} - (w_2)^2(\nu_{xx} + \nu_{yy} + \nu_{zz}) - 2a_2\rho(\rho_{xx} + \rho_{yy} + \rho_{zz}) - 2a_2({\rho_x}^2 + {\rho_y}^2 + {\rho_z}^2) = 0$$

- three coupled equations
- four independent variables x, y, z and t
- three dependent variables ρ, ϕ and ν
- vector field

$$\alpha = \eta^x \frac{\partial}{\partial x} + \eta^y \frac{\partial}{\partial y} + \eta^z \frac{\partial}{\partial z} + \eta^t \frac{\partial}{\partial t} + \varphi^\rho \frac{\partial}{\partial \rho} + \varphi^\phi \frac{\partial}{\partial \phi} + \varphi^\nu \frac{\partial}{\partial \nu}$$

Format for SYMMGRP.MAX

- variables

$$\begin{array}{rcl} x & \longmapsto & x[1] \\ y & \longmapsto & x[2] \\ z & \longmapsto & x[3] \\ t & \longmapsto & x[4] \end{array} \qquad \begin{array}{rcl} \rho & \longmapsto & u[1] \\ \phi & \longmapsto & u[2] \\ \nu & \longmapsto & u[3] \end{array} \tag{2}$$

- typical datafile (next page)
- variables to be eliminated

  v1 : u[1,[0,0,0,1]]

  v2 : u[2,[0,0,0,1]]

  v3 : u[3,[0,0,0,2]]

There are 69 determining equations (not shown)

The solution in the original variables

$$
\begin{aligned}
\eta^x &= k_1\, y + k_2 & \varphi^\rho &= 0 \\
\eta^y &= -k_1\, x + k_3 & \varphi^\phi &= k_7\, \mathrm{a}_1\, t^2 + k_8\, \mathrm{a}_1\, t + k_9 \\
\eta^z &= k_4 & \varphi^\nu &= -2\, k_7\, t - k_8 \\
\eta^t &= k_5
\end{aligned}
\tag{3}
$$

The eight infinitesimal generators are

$$
\begin{aligned}
G_1 &= \partial_x & G_5 &= y\partial_x - x\partial_y \\
G_2 &= \partial_y & G_6 &= \partial_\phi \\
G_3 &= \partial_z & G_7 &= a_1\, t\, \partial_\phi - \partial_\nu \\
G_4 &= \partial_t & G_8 &= a_1\, t^2\, \partial_\phi - 2t\, \partial_\nu
\end{aligned}
\tag{4}
$$

Typical Datafile

```
p : 4 $
q : 3 $
m : 3 $

parameters : [a1,a2,s1,s2,w1,w2] $
warnings : true $
sublisteqs : [all] $
info_given : false $
highest_derivatives : all $
subst_deriv_of_vi : false $

e1 : u[1,[0,0,0,1]]+w1*u[1,[0,0,1,0]]+(1/2)*(s1*(2*u[1,[1,0,0,0]]
     *u[2,[1,0,0,0]]+2*u[1,[0,1,0,0]]*u[2,[0,1,0,0]]+u[1]*u[2,[2,0,0,0]]
     +u[1]*u[2,[0,2,0,0]])+s2*(2*u[1,[0,0,1,0]]*u[2,[0,0,1,0]]
     +u[1]*u[2,[0,0,2,0]]));
e2 : u[2,[0,0,0,1]]+w1*u[2,[0,0,1,0]]-(1/2)*(s1*(u[1,[2,0,0,0]]/u[1]
     +u[1,[0,2,0,0]]/u[1]-u[2,[1,0,0,0]]^2 -u[2,[0,1,0,0]]^2 )
     +s2*(u[1,[0,0,2,0]]/u[1]-u[2,[0,0,1,0]]^2 ))+a1*u[3];
e3 : u[3,[0,0,0,2]]- v2^2 *(u[3,[2,0,0,0]]+u[3,[0,2,0,0]] +u[3,[0,0,2,0]])
     -2*a2*u[1]*(u[1,[2,0,0,0]]+u[1,[0,2,0,0]]+u[1,[0,0,2,0]])
     -2*a2*(u[1,[1,0,0,0]]^2 +u[1,[0,1,0,0]]^2 +u[1,[0,0,1,0]]^2 );

v1 : u[1,[0,0,0,1]];
v2 : u[2,[0,0,0,1]];
v3 : u[3,[0,0,0,2]];
```

Example 8: The Magneto-Hydro-Dynamics Eqs.

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + (\vec{v}\cdot\nabla)\rho + \rho\nabla\cdot\vec{v} &= 0 \\
\rho(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}) + \nabla(p + \frac{1}{2}\vec{H}^2) - (\vec{H}\cdot\nabla)\vec{H} &= \vec{0} \\
\frac{\partial \vec{H}}{\partial t} + (\vec{v}\cdot\nabla)\vec{H} + \vec{H}\nabla\cdot\vec{v} - (\vec{H}\cdot\nabla)\vec{v} &= \vec{0} \\
\nabla\cdot\vec{H} &= 0 \\
\frac{\partial}{\partial t}(\frac{p}{\rho^{\kappa}}) + (\vec{v}\cdot\nabla)(\frac{p}{\rho^{\kappa}}) &= 0
\end{aligned}$$

p pressure
ρ mass density
κ coefficient of viscosity
$\vec{v}$ fluid velocity
$\vec{H}$ magnetic field

- nine coupled scalar equations
- four independent variables x, y, z and t
- eight dependent variables $\rho, p, v_x, v_y, v_z, H_x, H_y$ and H_z
- vector field

$$\alpha = \eta^x \frac{\partial}{\partial x} + \eta^y \frac{\partial}{\partial y} + \eta^z \frac{\partial}{\partial z} + \eta^t \frac{\partial}{\partial t}$$

$$
\begin{aligned}
&+ \ \varphi^{\rho}\frac{\partial}{\partial \rho} + \varphi^{p}\frac{\partial}{\partial p} \\
&+ \ \varphi^{v_x}\frac{\partial}{\partial v_x} + \varphi^{v_y}\frac{\partial}{\partial v_y} + \varphi^{v_z}\frac{\partial}{\partial v_z} \\
&+ \ \varphi^{H_x}\frac{\partial}{\partial H_x} + \varphi^{H_y}\frac{\partial}{\partial H_y} + \varphi^{H_z}\frac{\partial}{\partial H_z}
\end{aligned}
$$

Format for SYMMGRP.MAX

- variables

$$
\begin{array}{rcl@{\qquad\qquad}rcl}
x & \longmapsto & x[1] & v_x & \longmapsto & u[3] \\
y & \longmapsto & x[2] & v_y & \longmapsto & u[4] \\
z & \longmapsto & x[3] & v_z & \longmapsto & u[5] \\
t & \longmapsto & x[4] & H_x & \longmapsto & u[6] \\
\rho & \longmapsto & u[1] & H_y & \longmapsto & u[7] \\
p & \longmapsto & u[2] & H_z & \longmapsto & u[8]
\end{array} \tag{5}
$$

- typical datafile (next page)
- variables to be eliminated vi $(i : 1 \ldots 9)$

There are 222 determining equations (not shown)

The solution in the original variables is

$$
\begin{aligned}
\eta^x &= k_2 + k_5\, t - k_8\, y - k_9\, z + k_{11}\, x \\
\eta^y &= k_3 + k_6\, t + k_8\, x - k_{10}\, z + k_{11}\, y \\
\eta^z &= k_4 + k_7\, t + k_9\, x + k_{10}\, y + k_{11}\, z \\
\eta^t &= k_1 + k_{12}\, t \\
\varphi^\rho &= -2\,(k_{11} - k_{12} - k_{13})\,\rho \\
\varphi^p &= 2\, k_{13}\, p \\
\varphi^{v_x} &= k_5 - k_8\, v_y - k_9\, v_z + (k_{11} - k_{12}) v_x \\
\varphi^{v_y} &= k_6 + k_8\, v_x - k_{10}\, v_z + (k_{11} - k_{12}) v_y \\
\varphi^{v_z} &= k_7 + k_9\, v_x + k_{10}\, v_y + (k_{11} - k_{12}) v_z \\
\varphi^{H_x} &= k_{13}\, H_x - k_8 H_y - k_9 H_z \\
\varphi^{H_y} &= k_{13}\, H_y + k_8 H_x - k_{10} H_z \\
\varphi^{H_z} &= k_{13}\, H_z + k_9 H_x + k_{10} H_y
\end{aligned}
$$

The thirteen infinitesimal generators are

$$
\begin{aligned}
G_1 &= \partial_t \\
G_2 &= \partial_x \\
G_3 &= \partial_y \\
G_4 &= \partial_z \\
G_5 &= t\partial_x + \partial_{v_x} \\
G_6 &= t\partial_y + \partial_{v_y} \\
G_7 &= t\partial_z + \partial_{v_z} \\
G_8 &= x\partial_y - y\partial x + v_x\partial_{v_y} - v_y\partial_{v_x} + H_x\partial_{H_y} - H_y\partial_{H_x} \\
G_9 &= y\partial_z - z\partial y + v_y\partial_{v_z} - v_z\partial_{v_y} + H_y\partial_{H_z} - H_z\partial_{H_y} \\
G_{10} &= z\partial_x - x\partial z + v_z\partial_{v_z} - v_x\partial_{v_z} + H_z\partial_{H_x} - H_x\partial_{H_z} \\
G_{11} &= x\partial_x + y\partial_y + z\partial_z - 2\rho\partial_\rho + v_x\partial_{v_x} + v_y\partial_{v_y} + v_z\partial_{v_z} \\
G_{12} &= t\partial_t + 2\rho\partial_\rho - (v_x\partial_{v_x} + v_y\partial_{v_y} + v_z\partial_{v_z}) \\
G_{13} &= 2\rho\partial_\rho + 2p\partial_p + H_x\partial_{H_x} + H_y\partial_{H_y} + H_z\partial_{H_z}
\end{aligned}
$$

Equations are invariant under:

- translations G_2 through G_4
- Galilean boosts G_5 through G_7
- rotations G_8 through G_{10}
- dilations G_{11} through G_{13}

Software Performance for $u_t + auu_x + u_{xxx} = 0$

Korteweg-de Vries	PC LIE Head	PDELIE Vafeades	SYM_DE Steinberg	SPDE Schwarz	SYMMGRP Hereman
Determining Equations	7	?	4 ‡	?	8
Solved Automatically	Y	Y	N	Y	N
Correct Solution	Y	Y †	Y	Y	Y
Errors	N	N	N	N	N
Trouble		parameter	parameter		
Notes		† $a = 1$	‡ 2 if $a = 1$		

Software Performance for $u_t - u^3 u_{xxx} = 0$

Harry Dym Eq	PC LIE Head	PDELIE Vafeades	SYM_DE Steinberg	SPDE Schwarz	SYMMGRP Hereman
Determining Equations	7	?	N	?	8
Solved Automatically	Y	Y	N	?	N
Correct Solution	Y	Y	N	?	Y
Errors	N	N	Y	?	N
Trouble			sym_oneterm cannot solve $\frac{DA2}{DU} U^6 = 0$		
Notes					

Software Performance for $u_t + 6uu_x + u_{xxx} + \frac{1}{2t}u = 0$

cylindrical KdV	PC LIE Head	PDELIE Vafeades	SYM_DE Steinberg	SPDE Schwarz	SYMMGRP Hereman
Determining Equations	7	?	3	?	8
Solved Automatically	Y	Y	N	?	N
Correct Solution	Y	N	Y	?	Y
Errors	N	N	N	?	N
Trouble					
Notes		incomplete			

Software Performance for $iu_t + u_{xx} + 2|u|^2 u = 0$

NL Schrödinger	PC LIE Head	PDELIE Vafeades	SYM_DE Steinberg	SPDE Schwarz	SYMMGRP Hereman
Determining Equations	15	?	10	40	20
Solved Automatically	Y	N	N	Y	N
Correct Solution	Y	N	Y	Y	Y
Errors	N	Y	N	N	N
Trouble		member atomic			
Notes					

Software Performance for

$$F[\rho(F_{\rho\rho} + F_{zz}) + F_\rho] - \rho(F_\rho^2 + F_z^2 - G_\rho^2 - G_z^2) = 0$$
$$F[\rho(G_{\rho\rho} + G_{zz}) + G_\rho] - 2\rho(F_\rho G_\rho + F_z G_z) = 0$$

Ernst Equations	PC LIE Head	PDELIE Vafeades	SYM_DE Steinberg	SPDE Schwarz	SYMMGRP Hereman
Determining Equations	15	?	23	21	25
Solved Automatically	Y	N	N	N ‡	N
Correct Solution	Y †	N	Y	N	Y
Errors	N	Y	N	Y	N
Trouble		member atomic		‡ info given	
Notes	† docheck twice			wrong in paper	

Software Performance for

$$
\begin{aligned}
i(\psi_t + w_1\ \psi_z) + \frac{1}{2}[s_1\ (\psi_{xx} + \psi_{yy}) + s_2\ \psi_{zz}] - a_1\ \nu\ \psi &= 0 \\
\nu_{tt} - (w_2)^2\ (\nu_{xx} + \nu_{yy} + \nu_{zz}) - a_2\ (|\psi|^2_{xx} + |\psi|^2_{yy} + |\psi|^2_{zz}) &= 0
\end{aligned}
$$

Karpman Eqs	PC LIE Head	PDELIE Vafeades	SYM_DE Steinberg	SPDE Schwarz	SYMMGRP Hereman
Determining Equations	74	?	? †	?	69
Solved Automatically	Y	?	?	?	N
Correct Solution	Y	?	?	?	Y
Errors	N	?	?	?	N
Trouble			† ran for hours on SUN station		
Notes					

Software Performance for

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + (\vec{v}\cdot\nabla)\rho + \rho\nabla\cdot\vec{v} &= 0 \\
\rho(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}) + \nabla(p + \frac{1}{2}\vec{H}^2) - (\vec{H}\cdot\nabla)\vec{H} &= \vec{0} \\
\frac{\partial \vec{H}}{\partial t} + (\vec{v}\cdot\nabla)\vec{H} + \vec{H}\nabla\cdot\vec{v} - (\vec{H}\cdot\nabla)\vec{v} &= \vec{0} \\
\nabla\cdot\vec{H} &= 0 \\
\frac{\partial}{\partial t}(\frac{p}{\rho^{\kappa}}) + (\vec{v}\cdot\nabla)(\frac{p}{\rho^{\kappa}}) &= 0
\end{aligned}
$$

MHD Equations	PC LIE Head	PDELIE Vafeades	SYM_DE Steinberg	SPDE Schwarz	SYMMGRP Hereman
Determining Equations	N †	?	?	160	222
Solved Automatically	?	?	?	Y ‡	N
Correct Solution	?	?	?	?	Y
Errors	?	?	?	?	N
Notes	† Out of Memory			‡ Used CRAY XMP	

Documentation and Examples

	PC LIE Head	PDELIE Vafeades	SYM_DE Steinberg	SPDE Schwarz	SYMMGRP Hereman
Source	Y	N	N	Y	Y
Available	PC	Macsyma	Macsyma	Reduce	CPC Lib
Usage Readme	1	1	1	1	1
Demo files	10	0	5	1	1
Worked Cases	10	5	12	6	12
Test Cases	30	12	20	30	50
Papers	1 †	1	1	2	1
Notes	† planned				

Tasks of the Programs

	PC LIE Head	PDELIE Vafeades	SYM_DE Steinberg	SPDE Schwarz	SYMMGRP Hereman
Determining eqs	Y	Y	Y	Y	Y
Solve (auto)	Y	Y	Y	Y	N
Solve (inter)	Y	Y	Y	Y	Y
Check coeff	Y	Y	Y	Y	Y
Generators	Y	Y	Y	Y	N
Commutators	Y	N	N	Y	N
Invariants	N	Y	Y †	N	N
Similarity sol	N	Y	N	N	N
General symm	N	Y	N ?	N	N
Other		type of symmetry	pol sol † lin system		large systems