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SYMBOLIC METHODS TO FIND EXACT SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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I. INTRODUCTION

- Goals:

  - easy construction of exact solutions
  - solitary wave solutions and solitons
  - investigate integrability

- Technique:

  - simplified version of Hirota's method
  - make method applicable to equations that have no bilinear form

- Implementation:

  - MACSYMA & MATHEMATICA
  - other symbolic manipulation programs

- Applications:

  - reaction-diffusion equations
  - 5th order evolution equations

II. SIMPLIFIED VERSION OF HIROTA'S METHOD

Hirota's method requires:

- a clever change of dependent variable
- the introduction of a bilinear differential operator
- a perturbation-like expansion

Example:

The Korteweg-de Vries equation

$$u_t + 6uu_x + u_{3x} = 0$$

Substitute the Laurent expansion

$$u(x,t) = f(x,t)^{\alpha} \sum_{k=0}^{\infty} u_k(x,t) f(x,t)^k$$

with $u_0(t,x) \neq 0$, α negative integer
$u_k(t,x)$ analytic in a neighborhood of the singular non-characteristic manifold $f(t,x) = 0$

Determine $\alpha = -2$ (leading order behavior)

Truncate expansion at constant level term

$$u(x,t) = f(x,t)^{-2}\left[u_0 + u_1\, f(x,t) + u_2(x,t)\, f(x,t)^2\right]$$

or with explicit forms of u_0, u_1 and with $u_2 = 0$

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} = 2\frac{(f f_{xx} - f_x^2)}{f^2}$$

Integrate with respect to x

$$f f_{xt} - f_x f_t + f f_{4x} - 4 f_x f_{3x} + 3 f_{2x}^2 = 0$$

Could be written in *bilinear form*

$$\left(D_x D_t + D_x^4\right)(f \cdot f) = 0$$

via Hirota's bilinear operator

$$D_x^m D_t^n (f \cdot g) = (\partial x - \partial x')^m (\partial t - \partial t')^n f(x,t) g(x',t')|_{x'=x, t'=t}$$

Make Hirota's technique applicable to equations that *cannot* be written in bilinear form
Leave Hirota's bilinear operators out

Write in general $\mathcal{N}(f, f) = 0$, with

$$\mathcal{N}(f,g) = (\mathcal{I}f)(\frac{\partial^2 g}{\partial x \partial t} + \frac{\partial^4 g}{\partial x^4}) - \frac{\partial f}{\partial x}\frac{\partial g}{\partial t} - 4\frac{\partial f}{\partial x}\frac{\partial^3 g}{\partial x^3} + 3\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 g}{\partial x^2}$$

$\mathcal{I}$ is the identity operator

Seek formal solution (book keeping parameter ϵ)

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x,t)$$

Perturbation scheme (equate powers in ϵ to zero)

$$\begin{aligned}
O(\epsilon^1) &: \mathcal{L}f^{(1)} = 0 \\
O(\epsilon^2) &: \mathcal{L}f^{(2)} = -\mathcal{N}(f^{(1)}, f^{(1)}) \\
O(\epsilon^3) &: \mathcal{L}f^{(3)} = -\mathcal{N}(f^{(1)}, f^{(2)}) - \mathcal{N}(f^{(2)}, f^{(1)}) \\
&\vdots \\
O(\epsilon^n) &: \mathcal{L}f^{(n)} = -\sum_{j=1}^{n-1} \mathcal{N}(f^{(j)}, f^{(n-j)})
\end{aligned}$$

where $\mathcal{L}$ denotes the linear differential operator

$$\mathcal{L}\bullet = \frac{\partial^2 \bullet}{\partial x \partial t} + \frac{\partial^4 \bullet}{\partial x^4}$$

N-soliton solution is then generated by

$$f^{(1)} = \sum_{i=1}^{N} f_i = \sum_{i=1}^{N} \exp(\theta_i) = \sum_{i=1}^{N} \exp\left(k_i\, x - \omega_i\, t + \delta_i\right)$$

with constant k_i, ω_i and δ_i

Determine the dispersion law (from level ϵ^1)

$$P(k_i, \omega_i) = -\omega_i k_i + {k_i}^4 = 0$$

Thus

$$\omega_i = k_i^3 \qquad\qquad i = 1, 2, ..., N$$

Compute RHS at level ϵ^2

$$-\sum_{i,j=1}^{N} 3k_i k_j^2 (k_i - k_j) f_i f_j = \sum_{1 \le i < j \le N} 3k_i k_j (k_i - k_j)^2 f_i f_j$$

Note: terms in f_i^2 drop out

Form of $f^{(2)}$ is determined

$$\begin{aligned} f^{(2)} &= \sum_{1 \le i < j \le N} a_{ij} f_i f_j \\ &= \sum_{1 \le i < j \le N} a_{ij} \exp[(k_i + k_j)x - (\omega_i + \omega_j)t + (\delta_i + \delta_j)] \end{aligned}$$

Compute LHS at level ϵ^2

$$\begin{aligned}\mathcal{L}f^{(2)} &= \sum_{1\le i<j\le N} P(k_i+k_j,\omega_i+\omega_j)\,a_{ij}\,f_i f_j \\ &= \sum_{1\le i<j\le N} 3k_ik_j(k_i+k_j)^2\,a_{ij}\,f_i f_j\end{aligned}$$

Equate LHS and RHS

$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2} \qquad\qquad 1 \le i < j \le N$$

Proceeding in a similar fashion with equation at order ϵ^3

Example: for $N = 3$ (three soliton solution)

$$\begin{aligned}f_2 &= a_{12}\exp(\theta_1+\theta_2) + a_{13}\exp(\theta_1+\theta_3) + a_{23}\exp(\theta_2+\theta_3) \\ &= a_{12}\exp\left[(k_1+k_2)x-(\omega_1+\omega_2)t+(\delta_1+\delta_2)\right] \\ &+ a_{13}\exp\left[(k_1+k_3)x-(\omega_1+\omega_3)t+(\delta_1+\delta_3)\right] \\ &+ a_{23}\exp\left[(k_2+k_3)x-(\omega_2+\omega_3)t+(\delta_2+\delta_3)\right]\end{aligned}$$

and

$$\begin{aligned}f_3 &= b_{123}\exp(\theta_1+\theta_2+\theta_3) \\ &= b_{123}\exp\left[(k_1+k_2+k_3)x-(\omega_1+\omega_2+\omega_3)t+(\delta_1+\delta_2+\delta_3)\right]\end{aligned}$$

with

$$b_{123} = a_{12}\, a_{13}\, a_{23} = \frac{(k_1 - k_2)^2\,(k_1 - k_3)^2\,(k_2 - k_3)^2}{(k_1 + k_2)^2\,(k_1 + k_3)^2\,(k_2 + k_3)^2}$$

Note: $f^{(3)}$ has no terms in $f_i^2 f_j \quad (i, j = 1, ..., N,\ i \neq j)$

Also: for $N = 3$, one has $f^{(n)} = 0$ for $n > 3$

The expansion truncates (set $\epsilon = 1$)

$$\begin{aligned} f &= 1 + \exp\theta_1 + \exp\theta_2 + \exp\theta_3 \\ &+ a_{12}\exp(\theta_1 + \theta_2) + a_{13}\exp(\theta_1 + \theta_3) + a_{23}\exp(\theta_2 + \theta_3) \\ &+ b_{123}\exp(\theta_1 + \theta_2 + \theta_3) \end{aligned}$$

Substitute the explicit form of $f(x,t)$ back into

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} = 2\frac{(f f_{xx} - f_x^2)}{f^2}$$

N-soliton solution for any $N > 3$ is constructed similarly

- Single soliton solution

$$\begin{aligned} f &= 1 + e^{\theta} \\ \theta &= kx - \omega t + \delta \end{aligned}$$

k, ω and δ are constants

$$P(k, -\omega) = -\omega k + k^4 = 0$$

Substituting f into

$$\begin{aligned} u(x,t) &= 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} \\ &= 2(\frac{f_{xx} f - f_x^2}{f^2}) \end{aligned}$$

Denote $k = 2K$, to get the solitary wave solution

$$u = 2K^2 \text{sech}^2 K(x - 4K^2 t + \delta)$$

- Two-soliton solution

$$
\begin{aligned}
f &= 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2} \\
\theta_i &= k_i x - \omega_i t + \delta_i
\end{aligned}
$$

$$
P(k_i, -\omega_i) = 0 \quad \text{or} \quad \omega_i = k_i^3 \quad (i = 1, 2)
$$

$$
\begin{aligned}
a_{12} &= -\frac{P(k_1 - k_2, -\omega_1 + \omega_2)}{P(k_1 + k_2, -\omega_1 - \omega_2)} \\
&= \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}
\end{aligned}
$$

Select

$$
\begin{aligned}
e^{\delta_i} &= \frac{c_i^2}{k_i} e^{k_i x - \omega_i t + \Delta_i} \\
\tilde{f} &= \frac{1}{4} f e^{-\frac{1}{2}(\tilde{\theta}_1 + \tilde{\theta}_2)} \\
\tilde{\theta}_i &= k_i x - \omega_i t + \Delta_i \\
c_i^2 &= \left(\frac{k_2 + k_1}{k_2 - k_1} \right) k_i
\end{aligned}
$$

One obtains upon return to u

$$
\begin{aligned}
u(x,t) &= \tilde{u}(x,t) = 2\frac{\partial^2 \ln \tilde{f}(x,t)}{\partial x^2} \\
&= \left(\frac{k_2^2 - k_1^2}{2}\right)\left(\frac{k_2^2 \mathrm{cosech}^2\frac{\tilde{\theta}_2}{2} + k_1^2 \mathrm{sech}^2\frac{\tilde{\theta}_1}{2}}{(k_2 \coth\frac{\tilde{\theta}_2}{2} - k_1 \tanh\frac{\tilde{\theta}_1}{2})^2}\right)
\end{aligned}
$$

- For the general N-soliton solution

$$
f = \sum_{\mu=0,1} \exp\left[\sum_{i<j}^{(N)} A_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \mu_i\theta_i\right]
$$

$$
a_{ij} = \exp A_{ij} = -\frac{P(k_i - k_j, -\omega_i + \omega_j)}{P(k_i + k_j, -\omega_i - \omega_j)}
$$

Additional condition for $P(D_x, D_t)$

$$
\begin{aligned}
S[P,n] &= \sum_{\sigma=\pm 1} P\left(\sum_{i=1}^{n} \sigma_i k_i, -\sum_{i=1}^{n} \sigma_i \omega_i\right) \\
&\quad \times \prod_{i<j}^{(n)} P(\sigma_i k_i - \sigma_j k_j, -\sigma_i\omega_i + \sigma_j\omega_j)\sigma_i\sigma_j = 0 \\
&\qquad \text{for } \; n = 2, \ldots, N
\end{aligned}
$$

III. THE FISHER EQUATION WITH CONVECTION

$$u_t + kuu_x - u_{xx} - u(1-u) = 0$$

k is the convection constant

Truncated Laurent expansion

$$u(x,t) = -\frac{2}{k}\frac{\partial \ln f(x,t)}{\partial x} = -\frac{2}{k}(\frac{f_x}{f})$$

transforms the PDE into

$$\mathcal{N}(f,f) = f(f_{xxx} + f_x - f_{xt}) + f_x(f_t - f_{xx} + \frac{2}{k}f_x) = 0$$

The nonlinear operator is given by

$$\mathcal{N}(f,g) = (\mathcal{I}f)(\frac{\partial^3 g}{\partial x^3} + \frac{\partial g}{\partial x} - \frac{\partial^2 g}{\partial x \partial t}) + (\frac{\partial f}{\partial x})(\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} + \frac{2}{k}\frac{\partial g}{\partial x})$$

Seek solution of type

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x,t)$$

Perturbation scheme:

$$\begin{aligned}
O(\epsilon^1) &: \mathcal{L}f^{(1)} = 0 \\
O(\epsilon^2) &: \mathcal{L}f^{(2)} = -\mathcal{N}(f^{(1)}, f^{(1)}) \\
O(\epsilon^3) &: \mathcal{L}f^{(3)} = -\mathcal{N}(f^{(1)}, f^{(2)}) - \mathcal{N}(f^{(2)}, f^{(1)}) \\
&\vdots \\
O(\epsilon^n) &: \mathcal{L}f^{(n)} = -\sum_{j=1}^{n-1} \mathcal{N}(f^{(j)}, f^{(n-j)})
\end{aligned}$$

with

$$\mathcal{L}\bullet = \frac{\partial^3 \bullet}{\partial x^3} + \frac{\partial \bullet}{\partial x} - \frac{\partial^2 \bullet}{\partial x \partial t}$$

Here

$$\omega_i = -(1 + k_i^2) \qquad i = 1, 2, ..., N$$

Consequently

$$\mathcal{L}f^{(2)} = -\sum_{i=1}^{N} k_i(1 + \frac{2}{k}k_i)f_i^2 - \sum_{1 \le i < j \le N} \frac{4}{k} k_i k_j (k_i + k_j) f_i f_j$$

Note presence of terms in f_i^2

We need to set $k_i = -\frac{k}{2}$

N-soliton solution for $N \geq 2$ does no longer exist

Leads to the case $N = 1$

$$f(x,t) = 1 + \exp\theta = 1 + \exp[-\frac{k}{2}x + \frac{1}{4}(4+k^2)t + \delta]$$

and

$$u(x,t) = \frac{\exp\theta}{c + \exp\theta} = \frac{\exp[-\frac{k}{2}x + \frac{1}{4}(4+k^2)t + \delta]}{c + \exp[-\frac{k}{2}x + \frac{1}{4}(4+k^2)t + \delta]}$$

Final solution in a more pleasing form

$$u(x,t) = \frac{1 + \tanh\frac{1}{2}\theta}{(1+c) + (1-c)\tanh\frac{1}{2}\theta}$$

Fisher equation without convection $(k = 0)$

Truncated Laurent expansion reveals the transformation

$$\begin{aligned} u(x,t) &= -6\frac{\partial^2 \ln f(x,t)}{\partial x^2} + \frac{6}{5}\frac{\partial \ln f(x,t)}{\partial t} \\ &= -6\frac{(f f_{xx} - f_x^2)}{f^2} + \frac{6}{5}(\frac{f_t}{f}) \end{aligned}$$

The quadratic equation in f and its derivatives is

$$\begin{aligned} \mathcal{N}(f,f) &= 5f(5f_{xxxx} + 5f_{xx} + f_{tt} - 6f_{xxt} - f_t) + 75f_{xx}^2 \\ &- 100f_x f_{xxx} - 25f_x^2 + f_t^2 - 30f_t f_{xx} + 60f_x f_{xt} = 0 \end{aligned}$$

Proceeding as above

$$u(x,t) = \frac{[1 - \tanh\frac{1}{2}\theta]^2}{[(1+c) - (1-d)\tanh\frac{1}{2}\theta]^2}$$

with either $d = c$ or $d = c + 4$, c any constant, and

$$\theta = \frac{1}{\sqrt{6}}x - \frac{5}{6}t + \delta$$

Note: that solution does not follow from the previous one for $k \longrightarrow 0$

IV. THE FITZHUGH-NAGUMO EQUATION WITH CONVECTION

$$u_t + kuu_x - u_{xx} - u(1-u)(a-u) = 0$$

k is convection constant

Substitute the Laurent expansion

$$u(x,t) = f(x,t)^{\alpha} \sum_{k=0}^{\infty} u_k(x,t) f(x,t)^k$$

Here, $\alpha = -1$, u_0 must satisfy

$$u_0^2 - ku_0 f_x - 2f_x^2 = 0$$

Resonances (u_r arbitrary)

$$\begin{aligned} r &= -1 \\ r &= 4 + k(\frac{u_0}{f_x}) = 2 + (\frac{u_0}{f_x})^2 \end{aligned}$$

For integer resonances

$$u_0(x,t) = \sqrt{m} f_x \qquad\qquad k = \frac{m-2}{\sqrt{m}}$$

m positive integer

Substitute the truncated Laurent expansion

$$u(x,t) = \sqrt{m}\frac{\partial \ln f(x,t)}{\partial x} + u_1(x,t) = \sqrt{m}\frac{f_x(x,t)}{f(x,t)} + u_1(x,t)$$

Collect power terms in $f(x,t) \rightarrow$
overdetermined system for $f(x,t)$ and $u_1(x,t)$:

$$f_t - (1+m) f_{xx} - \sqrt{m}\,[\frac{2}{m}(1+m)\,u_1 - 1 - a]\,f_x = 0$$

$$\begin{aligned} & f_{xt} - f_{xxx} + \frac{1}{\sqrt{m}}(m-2)\,u_1\,f_{xx} \\ & +[3{u_1}^2 - 2(1+a)u_1 + a + \frac{1}{\sqrt{m}}(m-2)\,(u_1)_x]\,f_x = 0 \end{aligned}$$

$$(u_1)_t + \frac{1}{\sqrt{m}}(m-2)u_1(u_1)_x - (u_1)_{xx} + u_1(1-u_1)(a-u_1) = 0$$

Trivial solutions $u_1 = 0, 1,$ or a

For $u_1 = 0$ ($u_1 = 1$ and $u_1 = a$ are similar)

Seek solution of type

$$f(x,t) = \sum_{i=1}^{N} exp(k_i x - \omega_i t + \delta_i)$$

Now $N = 2$, and

$$k_1 = \frac{1}{\sqrt{m}} \qquad \omega_1 = \frac{am-1}{m}$$

$$k_2 = \frac{a}{\sqrt{m}} \qquad \omega_2 = \frac{a(m-a)}{m}$$

Hence

$$f = c + exp[\frac{1}{\sqrt{m}}x + \frac{a(a-m)}{m}t + \delta_1] + exp[\frac{a}{\sqrt{m}}x + \frac{(1-am)}{m}t + \delta_2]$$

Returning to $u(x,t)$

$$u(x,t) = \frac{exp[\frac{1}{\sqrt{m}}x + \frac{(1-am)}{m}t + \delta_1] + a exp[\frac{a}{\sqrt{m}}x + \frac{a(a-m)}{m}t + \delta_2]}{c + exp[\frac{1}{\sqrt{m}}x + \frac{(1-am)}{m}t + \delta_1] + exp[\frac{a}{\sqrt{m}}x + \frac{a(a-m)}{m}t + \delta_2]}$$

This solution describes two coallescent wave fronts

Reduces for $m = 2$ to solution of FHN equation
without convection ($k = 0$)

V. FIFTH-ORDER EVOLUTION EQUATIONS

Class of equations

$$u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0$$

Special cases:

$\alpha = 30$	$\beta = 20$	$\gamma = 10$	Lax
$\alpha = 5$	$\beta = 5$	$\gamma = 5$	Sawada Kotera or Caudry−Dodd−Gibbon
$\alpha = 20$	$\beta = 25$	$\gamma = 10$	Kaup Kuperschmidt
$\alpha = 2$	$\beta = 6$	$\gamma = 3$	Ito

Substitute the Laurent expansion

$$u(x,t) = f(x,t)^{-2} \sum_{k=0}^{\infty} u_k(x,t) f(x,t)^k$$

Assume transformation of the form

$$u(x,t) = K \frac{\partial^2 \ln f(x,t)}{\partial x^2} = K \frac{(f f_{xx} - f_x^2)}{f^2}$$

Case that leads to N-soliton solution

$$K = \frac{60}{\beta+\gamma} \qquad \alpha = \frac{\gamma(\beta+\gamma)}{10}$$

Substitute and integrate with respect to x

$$\begin{aligned}&(\beta+\gamma)f^2[f_{xt}+f_{6x}] - f[(\beta+\gamma)f_t f_x + 6(\beta+\gamma)f_x f_{5x}\\&+15(\beta-3\gamma)f_{2x}f_{4x} - 20(\beta-2\gamma)f_{3x}^2]\\&+30(\beta-\gamma)[f_x^2 f_{4x} - 2f_x f_{2x} f_{3x} + f_{2x}^3] = 0\end{aligned}$$

One solitary wave solution

$$f(x,t) = 1 + exp(kx - \omega t + \delta) \qquad \omega = k^5$$

Thus

$$u(x,t) = \frac{15k^2}{\beta+\gamma}\,\mathrm{sech}^2\,\frac{1}{2}(kx - k^5 t + \delta)$$

Two soliton solution requires

$$\beta = 2\gamma \qquad \text{or} \qquad \beta = \gamma$$

Case 1: LAX equation $\quad (\beta = 2\gamma)$

$$
\begin{aligned}
& f^2[f_{xt} + f_{6x}] - f[f_t f_x + 6 f_x f_{5x} - 5 f_{2x} f_{4x}] \\
& +10[f_x^2 f_{4x} - 2 f_x f_{2x} f_{3x} + f_{2x}^3] = 0
\end{aligned}
$$

Note: This is a cubic equation!
Bilinear form consists of two equations

$$
\left(D_x D_\tau + D_x^4\right)(f \cdot f) = 0
$$

$$
\begin{aligned}
& \left(D_x D_t + D_x^6\right)(f \cdot f) + a\left(D_x^3 D_\tau + D_x^6\right)(f \cdot f) \\
& -\frac{5+6a}{3}\left(D_\tau^2 + D_\tau D_x^3\right)(f \cdot f) = 0
\end{aligned}
$$

with auxiliary time variable τ
Bilinear form is no longer needed

Write the cubic equation as

$$
f^2 \mathcal{L}(f) + f \mathcal{N}_1(f, f) + \mathcal{N}_2(f, f, f) = 0
$$

with

$$
\begin{aligned}
\mathcal{L}(f) &= f_{xt} + f_{6x} \\
\mathcal{N}_1(f, g) &= -(f_t g_x + 6 f_x g_{5x} - 5 f_{2x} g_{4x}) \\
\mathcal{N}_2(f, g, h) &= 10(f_x g_x h_{4x} - 2 f_x g_{2x} h_{3x} + f_{2x} g_{2x} h_{2x})
\end{aligned}
$$

Seek a solution of the form

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n \, f^{(n)}(x,t)$$

Perturbation scheme (equate powers in ϵ to zero)

$$\begin{aligned}
O(\epsilon^1) &: \ \mathcal{L}f^{(1)} = 0 \\
O(\epsilon^2) &: \ \mathcal{L}f^{(2)} = -2f^{(1)}\mathcal{L}f^{(1)} - \mathcal{N}_1(f^{(1)}, f^{(1)}) \\
O(\epsilon^3) &: \ \mathcal{L}f^{(3)} = -2f^{(1)}\mathcal{L}f^{(2)} - 2f^{(2)}\mathcal{L}f^{(1)} - {f^{(1)}}^2\mathcal{L}f^{(1)} \\
&\qquad\qquad - \mathcal{N}_1(f^{(1)}, f^{(2)}) - \mathcal{N}_1(f^{(2)}, f^{(1)}) \\
&\qquad\qquad - f^{(1)}\mathcal{N}_1(f^{(1)}, f^{(1)}) - \mathcal{N}_2(f^{(1)}, f^{(1)}, f^{(1)}) \\
&\ \vdots
\end{aligned}$$

N-soliton solution is then generated by

$$f^{(1)} = \sum_{i=1}^{N} f_i = \sum_{i=1}^{N} \exp(\theta_i) = \sum_{i=1}^{N} \exp\left(k_i \, x - \omega_i \, t + \delta_i\right)$$

with dispersion law (from level ϵ^1)

$$P(k_i, \omega_i) = -\omega_i k_i + {k_i}^6 = 0$$

or

$$\omega_i = k_i^5 \qquad\qquad i = 1, 2, ..., N$$

Then

$$\begin{aligned}\mathcal{L}f^{(2)} &= \sum_{1\le i<j\le N} P(k_i+k_j,\omega_i+\omega_j)\,a_{ij}\,f_if_j\\ &= \sum_{1\le i<j\le N} 5k_ik_j(k_i+k_j)^2(k_i^2+k_ik_j+k_j^2)\,a_{ij}\,f_if_j\end{aligned}$$

must balance

$$-\mathcal{N}_1(f^{(1)},f^{(1)}) = \sum_{1\le i<j\le N} 5k_ik_j(k_i-k_j)^2(k_i^2+k_ik_j+k_j^2)\,f_if_j$$

Hence

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}$$

N-soliton solution in analogous way

Case 2: SK or CDG equation $\quad (\beta = \gamma)$

$$\mathcal{N}(f,f) = f[f_{xt} + f_{6x}] - f_t f_x - 6 f_x f_{5x} + 15 f_{2x} f_{4x} - 10 f_{3x}^2 = 0$$

Note: This is a quadratic equation!
Bilinear representation

$$\left(D_x D_t + D_x^6\right)(f \cdot f) = 0$$

Easy to find but not needed!

Seek solution of the form (3-soliton)

$$\begin{aligned} f &= 1 + \exp\theta_1 + \exp\theta_2 + \exp\theta_3 \\ &+ a_{12}\exp(\theta_1+\theta_2) + a_{13}\exp(\theta_1+\theta_3) + a_{23}\exp(\theta_2+\theta_3) \\ &+ b_{123}\exp(\theta_1+\theta_2+\theta_3) \end{aligned}$$

Solve the perturbation scheme

Here

$$\begin{aligned} a_{ij} &= \frac{(k_i - k_j)^2 (k_i^2 - k_i k_j + k_j^2)}{(k_i + k_j)^2 (k_i^2 + k_i k_j + k_j^2)} = \frac{(k_i - k_j)^3 (k_i^3 + k_j^3)}{(k_i + k_j)^3 (k_i^3 - k_j^3)} \\ b_{123} &= a_{12} a_{13} a_{23} \end{aligned}$$

N-soliton solution in similar way