Los Alamos Days '92

SYMBOLIC SOFTWARE FOR SOLITON THEORY

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Symbolic Software Packages for Soliton Theory

- Painlevé Integrability Test of ODEs and PDEs
  - Painlevé Test for 3rd order equations by Hajee (Reduce, 1982)
  - ODE_Painlevé by Winternitz and Rand (Macsyma, 1986)
  - PDE_Painlevé by Hereman and Van de Bulck (Macsyma, 1987)
  - Painlevé program (parts) by Hlavatý (Reduce, 1986)

- Construction of Explicit Solitary Wave and Soliton Solutions
  - Solitary wave solutions by Hereman (Macsyma, 1990)
  - Solitons via Hirota's method by Hereman and Zhuang (Macsyma, 1990)

- Calculation of Lie-point and Generalized Symmetries
  - SPDE by Schwarz (Reduce, Scratchpad, 1986)
  - Symmetries via exterior differential forms by Kersten and Gragert (Reduce, 1987)
  - Lie-Bäcklund symmetries by Fedorova, Kornyak and Fushchich (Reduce, 1987)
  - Lie-point symmetries by Schwarzmeier and Rosenau (Macsyma, 1988)
  - Special symmetries by Mikhailov (Pascal, 1988)
  - LIE by Head (muMath, 1990)
  - PDELIE by Vafeades (Macsyma, 1990)
  - DEliA by Bocharov (Pascal, 1990)
  - SYM_DE by Steinberg (Macsyma, 1990)
  - SYMCAL by Reid (Macsyma, 1990)
  - SYMMGRP.MAX by Champagne, Hereman and Winternitz (1990)

Classes of Bilinear Equations

Major types of bilinear representations with examples:

Type I: Korteweg-de Vries equation

$$P(D_x, D_t)(f\cdot f) \overset{ex.}{=} (D_x D_t + D_x^4)(f\cdot f) = 0$$

Type II: modified Korteweg-de Vries equation

$$\begin{aligned} P_1(D_x, D_t)(f\cdot g) &\overset{ex.}{=} (D_t + D_x^3)(f\cdot g) = 0, \\ P_2(D_x, D_t)(f\cdot g) &\overset{ex.}{=} D_x^2(f\cdot g) = 0 \end{aligned}$$

Type III: sine-Gordon equation

$$\begin{aligned} P_1(D_x, D_t)(g\cdot f) &\overset{ex.}{=} (D_x D_t - 1)(g\cdot f) = 0, \\ P_2(D_x, D_t)(f\cdot f - g\cdot g) &\overset{ex.}{=} D_x D_t(f\cdot f - g\cdot g) = 0 \end{aligned}$$

Type IV: Nonlinear Schrödinger equation

$$\begin{aligned} P_1(D_x, D_t)(g\cdot f) &\overset{ex.}{=} (D_x^2 + iD_t)(g\cdot f) = 0, \\ P_2(D_x, D_t)(f\cdot f) - P_3(D_x, D_t)(g\cdot g^*) &\overset{ex.}{=} D_x^2(f\cdot f) - gg^* = 0 \end{aligned}$$

Type V: Benjamin-Ono equation

$$P(D_x, D_t)(f\cdot f^*) \overset{ex.}{=} (D_x^2 + iD_t)(f\cdot f^*) = 0$$

For equations of Type II, the three-soliton solution follows from

$$
\begin{aligned}
f &= 1 + i \exp \theta_1 + i \exp \theta_2 + i \exp \theta_3 \\
&- a_{12} \exp(\theta_1 + \theta_2) - a_{13} \exp(\theta_1 + \theta_3) - a_{23} \exp(\theta_2 + \theta_3) \\
&- i b_{123} \exp(\theta_1 + \theta_2 + \theta_3), \\
g &= 1 - i \exp \theta_1 - i \exp \theta_2 - i \exp \theta_3 \\
&- a_{12} \exp(\theta_1 + \theta_2) - a_{13} \exp(\theta_1 + \theta_3) - a_{23} \exp(\theta_2 + \theta_3) \\
&+ i b_{123} \exp(\theta_1 + \theta_2 + \theta_3).
\end{aligned}
$$

For equations of Types I, III through V other forms for f and g are needed

Example 1 - Macsyma
Soliton Solutions of Nonlinear PDEs

- Hirota's Direct Method
  allows to construct exact soliton solutions of
  - nonlinear evolution equations
  - wave equations
  - coupled systems
- Test conditions for existence of soliton solutions
- Examples:
  - Korteweg-de Vries equation (KdV)

$$u_t + 6uu_x + u_{3x} = 0$$

  - Kadomtsev-Petviashvili equation (KP)

$$(u_t + 6uu_x + u_{3x})_x + 3u_{2y} = 0$$

  - Sawada-Kotera equation (SK)

$$u_t + 45u^2u_x + 15u_xu_{2x} + 15uu_{3x} + u_{5x} = 0$$

Hirota's Method

Korteweg-de Vries equation

$$u_t + 6uu_x + u_{3x} = 0$$

Substitute

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}$$

Integrate with respect to x

$$f f_{xt} - f_x f_t + f f_{4x} - 4 f_x f_{3x} + 3 f_{2x}^2 = 0$$

Bilinear form

$$B(f \cdot f) \stackrel{\text{def}}{=} \left(D_x D_t + D_x^4\right)(f \cdot f) = 0$$

Introduce the bilinear operator

$$D_x^m\, D_t^n (f \cdot g) = (\partial x - \partial x')^m (\partial t - \partial t')^n f(x,t)\, g(x',t')|_{x'=x, t'=t}$$

Use the expansion

$$f = 1 + \sum_{n=1}^{\infty} \epsilon^n f_n$$

Substitute f into the bilinear equation

Collect powers in ϵ (book keeping parameter)

$$O(\epsilon^0) \ : \ B(1 \cdot 1) = 0$$

$$O(\epsilon^1) \ : \ B(1 \cdot f_1 + f_1 \cdot 1) = 0$$

$$\begin{aligned}
O(\epsilon^2) &: B(1{\cdot}f_2 + f_1{\cdot}f_1 + f_2{\cdot}1) = 0 \\
O(\epsilon^3) &: B(1{\cdot}f_3 + f_1{\cdot}f_2 + f_2{\cdot}f_1 + f_3{\cdot}1) = 0 \\
O(\epsilon^4) &: B(1{\cdot}f_4 + f_1{\cdot}f_3 + f_2{\cdot}f_2 + f_3{\cdot}f_1 + f_4{\cdot}1) = 0 \\
O(\epsilon^n) &: B(\sum_{j=0}^{n} f_j{\cdot}f_{n-j}) = 0 \qquad \text{with } f_0 = 1
\end{aligned}$$

If the original PDE admits a N-soliton solution
then the expansion will truncate at level $n = N$ provided

$$f_1 = \sum_{i=1}^{N} \exp(\theta_i) = \sum_{i=1}^{N} \exp\left(k_i\, x - \omega_i\, t + \delta_i\right)$$

k_i, ω_i and δ_i are constants

Dispersion law

$$\omega_i = k_i^3 \qquad (i = 1, 2, ..., N)$$

Consider the case N=3
Terms generated by $B(f_1, f_1)$ justify

$$\begin{aligned}
f_2 &= a_{12}\, \exp(\theta_1 + \theta_2) + a_{13}\, \exp(\theta_1 + \theta_3) + a_{23}\, \exp(\theta_2 + \theta_3) \\
&= a_{12}\, \exp\left[(k_1 + k_2)\, x - (\omega_1 + \omega_2)\, t + (\delta_1 + \delta_2)\right] \\
&+ a_{13}\, \exp\left[(k_1 + k_3)\, x - (\omega_1 + \omega_3)\, t + (\delta_1 + \delta_3)\right] \\
&+ a_{23}\, \exp\left[(k_2 + k_3)\, x - (\omega_2 + \omega_3)\, t + (\delta_2 + \delta_3)\right]
\end{aligned}$$

Calculate the constants a_{12}, a_{13} and a_{23}

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2} \qquad i, j = 1, 2, 3$$

$B(f_1 \cdot f_2 + f_2 \cdot f_1)$ motivates

$$\begin{aligned} f_3 &= b_{123} \exp(\theta_1 + \theta_2 + \theta_3) \\ &= b_{123} \exp\left[(k_1+k_2+k_3)x-(\omega_1+\omega_2+\omega_3)t+(\delta_1+\delta_2+\delta_3)\right] \end{aligned}$$

with

$$b_{123} = a_{12}\, a_{13}\, a_{23} = \frac{(k_1 - k_2)^2 (k_1 - k_3)^2 (k_2 - k_3)^2}{(k_1 + k_2)^2 (k_1 + k_3)^2 (k_2 + k_3)^2}$$

Subsequently, $f_i = 0$ for $i > 3$

Set $\epsilon = 1$

$$\begin{aligned} f &= 1 + \exp\theta_1 + \exp\theta_2 + \exp\theta_3 \\ &+ a_{12} \exp(\theta_1 + \theta_2) + a_{13} \exp(\theta_1 + \theta_3) + a_{23} \exp(\theta_2 + \theta_3) \\ &+ b_{123} \exp(\theta_1 + \theta_2 + \theta_3) \end{aligned}$$

Return to the original $u(x, t)$

$$u(x, t) = 2 \frac{\partial^2 \ln f(x, t)}{\partial x^2}$$

Example 2 - Macsyma
Solitary Wave Solutions

- Korteweg-de Vries equation and generalizations

$$u_t + a u^n u_x + u_{xxx} = 0, \quad n \in \mathbb{N}$$

$$u(x,t) = \left\{ \frac{c(n+1)(n+2)}{2a} \operatorname{sech}^2 \left[\frac{n}{2} \sqrt{c} (x - ct) + \delta \right] \right\}^{\frac{1}{n}}$$

- Burgers equation

$$u_t + a u u_x - u_{xx} = 0$$

$$u(x,t) = \frac{c}{a} \left\{ 1 - \tanh \left[\frac{c}{2} (x - ct) + \delta \right] \right\}$$

- Fisher equation and generalizations

$$u_t - u_{xx} - u(1 - u^n) = 0, \quad n \in \mathbb{N}$$

$$u(x,t) = \left\{ \frac{1}{2} \left[1 - \tanh \left[\frac{n}{2\sqrt{2n+4}} (x - \frac{(n+4)}{\sqrt{2n+4}} t) + \delta \right] \right] \right\}^{\frac{2}{n}}$$

- Fitzhugh-Nagumo equation

$$u_t - u_{xx} + u(1-u)(a-u) = 0$$

$$u(x,t) = \frac{a}{2}\left\{1 + \tanh\left[\frac{a}{2\sqrt{2}}(x - \frac{(2-a)}{\sqrt{2}}t) + \delta\right]\right\}$$

- Kuramoto-Sivashinski equation

$$u_t + uu_x + au_{xx} + bu_{xxxx} = 0$$

$$\begin{aligned} u(x,t) &= c + \frac{165ak}{19}\left\{\tanh^3\left[\frac{k(x-ct)}{2} + \delta\right]\right\} \\ &- \frac{135ak}{19}\left\{\tanh\left[\frac{k(x-ct)}{2} + \delta\right]\right\} \end{aligned}$$

with $k = \sqrt{\frac{11a}{19b}}$

$$\begin{aligned} u(x,t) &= c - \frac{15ak}{19}\left\{\tanh^3\left[\frac{k(x-ct)}{2} + \delta\right]\right\} \\ &+ \frac{45ak}{19}\left\{\tanh\left[\frac{k(x-ct)}{2} + \delta\right]\right\} \end{aligned}$$

with $k = \sqrt{\frac{-a}{19b}}$

- Harry Dym equation

$$u_t + (1-u)^3 u_{xxx} = 0$$

$$\begin{aligned} u(x,t) &= \operatorname{sech}^2\left[\frac{1}{2}\sqrt{c}\,[x - ct + \delta(x,t)]\right] \\ \delta(x,t) &= \frac{2}{\sqrt{c}}\tanh\left[\frac{\sqrt{c}}{2}\,[x - ct + \delta(x,t)]\right] \end{aligned}$$

- sine-Gordon equation

$$u_{tt} - u_{xx} - \sin u = 0$$

$$u(x,t) = 4\arctan\left\{\exp\left[\frac{1}{\sqrt{-c}}(x-ct) + \delta\right]\right\}$$

• Coupled Korteweg-de Vries equations

$$u_t - a(6uu_x + u_{xxx}) - 2b\, vv_x = 0,$$
$$v_t + 3uv_x + v_{xxx} = 0$$

$$\begin{aligned} u(x,t) &= 2\, c \,\mathrm{sech}^2\left[\sqrt{c}(x-ct)+\delta\right], \\ v(x,t) &= \pm c\sqrt{\frac{-2(4a+1)}{b}}\,\mathrm{sech}\left[\sqrt{c}(x-ct)+\delta\right], \end{aligned}$$

$$\begin{aligned} u(x,t) &= c\,\mathrm{sech}^2\left[\frac{1}{2}\sqrt{c}(x-ct)+\delta\right] \\ v(x,t) &= \frac{3}{\sqrt{6|b|}}\,u(x,t) = \frac{3\,c}{\sqrt{6|b|}}\,\mathrm{sech}^2\left[\frac{1}{2}\sqrt{c}(x-ct)+\delta\right] \end{aligned}$$

- A class of generalized KdV equations

$$u_t + (a + bu^m)u^m u_x + u_{3x} = 0$$

with $a, b \in \mathbb{R}; m \in \mathbb{Q}$

  - CASE 1: $a \neq 0, b = 0:$

    $$u(x,t) = \{\frac{c(m+2)(m+1)}{2a}\text{sech}^2[\frac{m}{2}\sqrt{c}(x-ct)+\frac{\Delta}{2}]\}^{\frac{1}{m}}$$

    with arbitrary velocity c

  - CASE 2: $b \neq 0:$

    $$u(x,t) = \{\frac{-a(2m+1)}{2b(m+2)}(1-\tanh[\frac{m}{2}\sqrt{c}(x-ct)+\frac{\Delta}{2}])\}^{\frac{1}{m}}$$

    with $c = -\frac{a^2(2m+1)}{b(m+1)(m+2)^2}$

Example 4 - Macsyma Lie-point Symmetries

- System of m differential equations of order k

$$\Delta^i(x, u^{(k)}) = 0, \quad i = 1, 2, ..., m$$

with p independent and q dependent variables

$$\begin{aligned} x &= (x_1, x_2, ..., x_p) \in \mathbb{R}^p \\ u &= (u^1, u^2, ..., u^q) \in \mathbb{R}^q \end{aligned}$$

- The group transformations have the form

$$\tilde{x} = \Lambda_{group}(x, u), \quad \tilde{u} = \Omega_{group}(x, u)$$

where the functions Λ_{group} and Ω_{group} are to be determined

- Look for the Lie algebra $\mathcal{L}$ realized by the vector field

$$\alpha = \sum_{i=1}^{p} \eta^i(x, u) \frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \varphi_l(x, u) \frac{\partial}{\partial u^l}$$

Procedure for finding the coefficients

- Construct the k^{th} prolongation $\mathrm{pr}^{(k)}\alpha$ of the vector field α
- Apply it to the system of equations
- Request that the resulting expression vanishes on the solution set of the given system

$$\mathrm{pr}^{(k)}\alpha\Delta^i\,|_{\Delta^j=0} \qquad i,j=1,...,m$$

- This results in a system of linear homogeneous PDEs for η^i and φ_l, with independent variables x and u (*determining equations*)
- Procedure thus consists of two major steps:

deriving the determining equations
solving the determining equations

Procedure for Computing the Determining Equations

- Use multi-index notation $J = (j_1, j_2, ..., j_p) \in \mathbb{N}^p$, to denote partial derivatives of u^l

$$u^l_J \equiv \frac{\partial^{|J|} u^l}{\partial {x_1}^{j_1} \partial {x_2}^{j_2} ... \partial {x_p}^{j_p}} \,,$$

where $|J| = j_1 + j_2 + ... + j_p$

- $u^{(k)}$ denotes a vector whose components are all the partial derivatives of order 0 up to k of all the u^l

- Steps:

(1) Construct the k^{th} prolongation of the vector field

$$\mathrm{pr}^{(k)}\alpha = \alpha + \sum_{l=1}^{q} \sum_{J} \psi_l^J(x, u^{(k)}) \frac{\partial}{\partial u_J^l}, \quad 1 \leq |J| \leq k$$

The coefficients ψ_l^J of the first prolongation are:

$$\psi_l^{J_i} = D_i \varphi_l(x, u) - \sum_{j=1}^{p} u^l_{J_j} D_i \eta^j(x, u),$$

where J_i is a $p-$tuple with 1 on the i^{th} position and zeros elsewhere

D_i is the total derivative operator

$$D_i = \frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \sum_{J} u^l_{J+J_i} \frac{\partial}{\partial u^l_J}, \quad 0 \leq |J| \leq k$$

Higher order prolongations are defined recursively:

$$\psi_l^{J+J_i} = D_i \psi_l^J - \sum_{j=1}^{p} u^l_{J+J_j} D_i \eta^j(x,u), \qquad |J| \geq 1$$

(2) Apply the prolonged operator $\mathrm{pr}^{(k)}\alpha$ to each equation $\Delta^i(x, u^{(k)}) = 0$

Require that $\mathrm{pr}^{(k)}\alpha$ vanishes on the solution set of the system

$$\mathrm{pr}^{(k)}\alpha \; \Delta^i \mid_{\Delta^j=0} \; = 0 \quad i,j = 1, ..., m$$

(3) Choose m components of the vector $u^{(k)}$, say $v^1, ..., v^m$, such that:

(a) Each v^i is equal to a derivative of a u^l $(l = 1, ..., q)$ with respect to at least one variable x_i $(i = 1, ..., p)$.

(b) None of the v^i is the derivative of another one in the set.

(c) The system can be solved algebraically for the v^i in terms of the remaining components of $u^{(k)}$, which we denoted by w:

$$v^i = S^i(x, w), \quad i = 1, ..., m.$$

(d) The derivatives of v^i,

$$v^i_J = D_J S^i(x, w),$$

where $D_J \equiv D_1^{j_1} D_2^{j_2} ... D_p^{j_p}$, can all be expressed in terms of the components of w and their derivatives, without ever reintroducing the v^i or their derivatives.

For instance, for a system of evolution equations

$$u^i_t(x_1, ..., x_{p-1}, t) = F^i(x_1, ..., x_{p-1}, t, u^{(k)}), \quad i = 1, ..., m,$$

where $u^{(k)}$ involves derivatives with respect to the variables x_i but not t, choose $v^i = u^i_t$.

(4) Eliminate all v^i and their derivatives from the expression prolonged vector field, so that all the remaining variables are independent

(5) Obtain the determining equations for $\eta^i(x, u)$ and $\varphi_l(x, u)$ by equating to zero the coefficients of the remaining independent derivatives u^l_J.