Conserved Densities and Generalized Symmetries of Nonlinear Differential-Difference Equations

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NEEDS 2001
Newton Institute for Mathematical Sciences
Cambridge, United Kingdom

July 24-31, 2001

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Research supported in part by NSF under Grant CCR-9901929

OUTLINE

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- Algorithm for conserved densities

Differential-difference Equations (DDEs)

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- **Purpose**

  Design and implement algorithms to compute polynomial conservation laws and generalized symmetries of nonlinear systems of PDEs and differential-difference equations (DDEs).

- **Motivation**

  – Conservation laws describe the conservation of fundamental physical quantities (linear momentum, energy, etc.).

  Compare with constants of motion (linear momentum, energy) in mechanics.

  – Conservation laws provide a method to study quantitative and qualitative properties of equations and their solutions, e.g. Hamiltonian structures.

  – Conserved densities can be used to test numerical integrators.

  – For PDEs and DDEs, the existence of a sufficiently large (in principal infinite) number of conservation laws or symmetries assures complete **integrability**.

  – Conserved densities and symmetries aid in finding the recursion operator (which guarantees the existence of infinitely many symmetries).

Partial Differential Equations (PDEs): Brief Review

- **Given: Nonlinear system of evolution equations**

$$\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ..., \mathbf{u}_{mx})$$

in a (single) space variable x and time t, and with

$$\mathbf{u} = (u_1, u_2, ..., u_n), \quad \mathbf{F} = (F_1, F_2, ..., F_n).$$

Notation:

$$\mathbf{u}_t = \frac{\partial \mathbf{u}}{\partial t}, \quad \mathbf{u}_{mx} = \mathbf{u}^{(m)} = \frac{\partial \mathbf{u}}{\partial x^m}.$$

$\mathbf{F}$ is polynomial in $\mathbf{u}, \mathbf{u}_x, ..., \mathbf{u}_{mx}$.

PDEs of higher order in t should be recast as a first-order system.

Prototypical Examples

The Korteweg-de Vries (KdV) equation:

$$u_t + uu_x + u_{3x} = 0.$$

The Boussinesq (wave) equation:

$$u_{tt} - u_{2x} + 3uu_{2x} + 3{u_x}^2 + \alpha u_{4x} = 0,$$

written as a first-order system (v auxiliary variable):

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + u_x - 3uu_x - \alpha u_{3x} &= 0. \end{aligned}$$

- **Dilation invariance**

The KdV equation, $u_t + uu_x + u_{3x} = 0$, has scaling symmetry

$$(t, x, u) \rightarrow (\lambda^{-3}t, \lambda^{-1}x, \lambda^2 u).$$

The Boussinesq system

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + u_x - 3uu_x - \alpha u_{3x} &= 0, \end{aligned}$$

is not scaling invariant (u_x and u_{3x} are conflicting terms).

If one introduces an auxiliary parameter β, then

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + \beta u_x - 3uu_x - \alpha u_{3x} &= 0, \end{aligned}$$

has scaling symmetry:

$$(x, t, u, v, \beta) \rightarrow (\lambda^{-1}x, \lambda^{-2}t, \lambda^2 u, \lambda^3 v, \lambda^2 \beta).$$

- **Conservation Laws**

$$\mathrm{D}_t \rho + \mathrm{D}_x J = 0,$$

with conserved density ρ and flux J.

Both are polynomial in $\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, \mathbf{u}_{3x}, \ldots$.

$$P = \int_{-\infty}^{+\infty} \rho \, dx = \text{constant}$$

if J vanishes at infinity.

Examples

For the Korteweg-de Vries (KdV) equation, $u_t + uu_x + u_{3x} = 0$, the first few conserved densities (and fluxes) are:

$$\rho_1 = u, \qquad D_t(u) + D_x(\frac{u^2}{2} + u_{2x}) = 0.$$

$$\rho_2 = u^2, \qquad D_t(u^2) + D_x(\frac{2u^3}{3} + 2uu_{2x} - {u_x}^2) = 0.$$

$$\rho_3 = u^3 - 3{u_x}^2,$$

$$D_t\left(u^3 - 3{u_x}^2\right) + D_x\left(\frac{3}{4}u^4 - 6u{u_x}^2 + 3u^2u_{2x} + 3{u_{2x}}^2 - 6u_xu_{3x}\right) = 0.$$

$\vdots$

$$\rho_6 = u^6 - 60u^3{u_x}^2 - 30{u_x}^4 + 108u^2{u_{2x}}^2$$

$$+\frac{720}{7}{u_{2x}}^3 - \frac{648}{7}u{u_{3x}}^2 + \frac{216}{7}{u_{4x}}^2.$$

The Boussinesq system:

$$\rho_1 = u, \qquad \rho_2 = v,$$
$$\rho_3 = uv, \qquad \rho_4 = \beta u^2 - u^3 + v^2 + \alpha {u_x}^2.$$

(then set $\beta = 1$).

- **Generalized Symmetries.**

$$\mathbf{G}(x, t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, \ldots)$$

with $\mathbf{G} = (G_1, G_2, \ldots, G_n)$ is a *symmetry* iff it leaves the PDE invariant for the replacement $\mathbf{u} \to \mathbf{u} + \epsilon\mathbf{G}$ within order ϵ. i.e.

$$\mathrm{D}_t(\mathbf{u} + \epsilon\mathbf{G}) = \mathbf{F}(\mathbf{u} + \epsilon\mathbf{G})$$

must hold up to order ϵ on the solutions of PDE.

Consequently, $\mathbf{G}$ must satisfy the linearized equation

$$\mathrm{D}_t\mathbf{G} = \mathbf{F}'(\mathbf{u})[\mathbf{G}] = \frac{\partial}{\partial\epsilon}\mathbf{F}(\mathbf{u} + \epsilon\mathbf{G})|_{\epsilon=0}.$$

where $\mathbf{F}'$ is the Fréchet derivative of $\mathbf{F}$.

Here $\mathbf{u}$ is replaced by $\mathbf{u} + \epsilon\mathbf{G}$, and $\mathbf{u}_{nx}$ by $\mathbf{u}_{nx} + \epsilon\mathrm{D}_x^n\mathbf{G}$.

Example

For the KdV equation, $u_t = 6uu_x + u_{3x}$, the first few generalized symmetries are:

$$
\begin{aligned}
G^{(1)} &= u_x, \\
G^{(2)} &= 6uu_x + u_{3x}, \\
G^{(3)} &= 30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x}, \\
G^{(4)} &= 140u^3u_x + 70u_x^3 + 280uu_xu_{2x} + 70u^2u_{3x} \\
&\quad +70u_{2x}u_{3x} + 42u_xu_{4x} + 14uu_{5x} + u_{7x}.
\end{aligned}
$$

- **Recursion Operators.**

A *recursion operator* $\mathbf{\Phi}$ connects two consecutive symmetries

$$\mathbf{G}^{(i+1)} = \mathbf{\Phi}\mathbf{G}^{(i)}.$$

For n-component systems, $\mathbf{\Phi}$ is an $n \times n$ matrix.

Defining equation for $\mathbf{\Phi}$:

$$\mathrm{D}_t\mathbf{\Phi} + [\mathbf{\Phi}, \mathbf{F}'(u)] = \frac{\partial \mathbf{\Phi}}{\partial t} + \mathbf{\Phi}'[\mathbf{F}] + \mathbf{\Phi} \circ \mathbf{F}'(u) - \mathbf{F}'(u) \circ \mathbf{\Phi} = 0,$$

where [,] means commutator, $\circ$ stands for composition, and $\mathbf{\Phi}'[\mathbf{F}]$ is the variational derivative of $\mathbf{\Phi}$.

Example

The recursion operator for the KdV equation:

$$\Phi = \mathrm{D}_x^2 + 2u + 2\mathrm{D}_x u \mathrm{D}_x^{-1} = \mathrm{D}_x^2 + 4u + 2u_x\mathrm{D}_x^{-1},$$

where D_x^{-1} is the integration operator.

For example,

$$\begin{aligned}
\Phi u_x &= (\mathrm{D}_x^2 + 2u + 2\mathrm{D}_x u \mathrm{D}_x^{-1}) u_x = 6uu_x + u_{3x}, \\
\Phi(6uu_x + u_{3x}) &= (\mathrm{D}_x^2 + 2u + 2\mathrm{D}_x u \mathrm{D}_x^{-1})(6uu_x + u_{3x}) \\
&= 30u^2 u_x + 20u_x u_{2x} + 10uu_{3x} + u_{5x}.
\end{aligned}$$

- **Key observations**:

  Conserved densities, generalized symmetries and recursion operators are invariant under the dilation (scaling) symmetry of the given PDE.

  The dilation invariance allows one to design algorithms to compute densities, symmetries, and recursion operators.

- **Example: Algorithm for Conserved Densities of PDEs.**

  1). Determine weights (scaling properties) of variables and auxiliary parameters.

  2). Construct the form of the density (find monomial building blocks).

  3). Determine the constant coefficients.

  Example: Density of **rank** 6 for the KdV equation.

  Step 1: **Compute the weights (dilation symmetry).**

  The *weight*, w, of a variable is by definition the number of x-derivatives the variable corresponds to.

  The *rank* of a monomial is its total weight in terms of x-derivatives.

  Set $w(\mathrm{D}_x) = 1$ or $w(x) = -1$ and require that all terms in

$$u_t + uu_x + u_{3x} = 0.$$

have the same rank. Hence,

$$w(u) + w(\mathrm{D}_t) = 2w(u) + 1 = w(u) + 3.$$

Solve the linear system: $w(u) = 2$, $w(\mathrm{D}_t) = 3$, so, $w(t) = -3$.

$$(t, x, u) \rightarrow (\lambda^{-3}t, \lambda^{-1}x, \lambda^2 u).$$

Step 2: Determine the form of the density.

List all possible powers of u, up to rank 6 : $\quad [u, u^2, u^3]$.

Introduce x derivatives to 'complete' the rank.

u has weight 2, introduce D_x^4.

u^2 has weight 4, introduce D_x^2.

u^3 has weight 6, no derivative needed.

Apply the D_x derivatives.
Remove terms of the form $D_x u_{px}$, or D_x up to terms kept prior in the list.

$$\begin{aligned} [u_{4x}] &\rightarrow [\,] \quad \text{empty list.} \\ [{u_x}^2, u u_{2x}] &\rightarrow [{u_x}^2] \quad \text{since } u u_{2x} = (u u_x)_x - {u_x}^2. \\ [u^3] &\rightarrow [u^3]. \end{aligned}$$

Linearly combine the 'building blocks':

$$\rho = c_1 u^3 + c_2 {u_x}^2.$$

Step 3: Determine the coefficients c_i.

Compute $D_t \rho = 3c_1 u^2 u_t + 2c_2 u_x u_{xt}$.

Replace u_t by $-(u u_x + u_{3x})$ and u_{xt} by $-(u u_x + u_{3x})_x$.

Integrate the result, E, with respect to x. To avoid integration by parts, apply the Euler operator (variational derivative)

$$
\begin{aligned}
L_u &= \sum_{k=0}^{m} (-\mathrm{D}_x)^k \frac{\partial}{\partial u_{kx}} \\
&= \frac{\partial}{\partial u} - \mathrm{D}_x(\frac{\partial}{\partial u_x}) + \mathrm{D}_x^2(\frac{\partial}{\partial u_{2x}}) + \cdots + (-1)^m \mathrm{D}_x^m(\frac{\partial}{\partial u_{mx}}).
\end{aligned}
$$

to E of order m.

If $L_u(E) = 0$ immediately, then E is a total x-derivative.

If $L_u(E) \neq 0$, the remaining expression must vanish identically.

$$
\begin{aligned}
D_t \rho =& -D_x[\frac{3}{4} c_1 u^4 - (3c_1 - c_2) u u_x^2 + 3c_1 u^2 u_{2x} \\
& - c_2 {u_{2x}}^2 + 2c_2 u_x u_{3x}] - (3c_1 + c_2) {u_x}^3.
\end{aligned}
$$

The non-integrable term must vanish.

So, $c_1 = -\frac{1}{3} c_2$. Set $c_2 = -3$, hence, $c_1 = 1$.

Result:

$$\rho = u^3 - 3{u_x}^2.$$

Expression [...] yields

$$J = \frac{3}{4} u^4 - 6u{u_x}^2 + 3u^2 u_{2x} + 3{u_{2x}}^2 - 6u_x u_{3x}.$$

Generalized symmetries and recursion operators can be constructed in a similiar way!

	Continuous Case (PDEs)	Semi-discrete Case (DDEs)
System	$\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ...)$	$\dot{\mathbf{u}}_n = \mathbf{F}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$
Conservation Law	$\mathrm{D}_t\rho + \mathrm{D}_x J = 0$	$\dot{\rho}_n + J_{n+1} - J_n = 0$
Symmetry	$\mathrm{D}_t\mathbf{G} = \mathbf{F}'(\mathbf{u})[\mathbf{G}] = \frac{\partial}{\partial\epsilon}\mathbf{F}(\mathbf{u} + \epsilon\mathbf{G})\vert_{\epsilon=0}$	$\mathrm{D}_t\mathbf{G} = \mathbf{F}'(\mathbf{u}_n)[\mathbf{G}] = \frac{\partial}{\partial\epsilon}\mathbf{F}(\mathbf{u}_n + \epsilon\mathbf{G})\vert_{\epsilon=0}$

Table 1: Conservation Laws and Symmetries

	KdV Equation	Volterra Lattice
Equation	$u_t = 6uu_x + u_{3x}$	$\dot{u}_n = u_n\,(u_{n+1} - u_{n-1})$
Densities	$\rho = u, \quad \rho = u^2$ $\rho = u^3 - \frac{1}{2}u_x^2$	$\rho_n = u_n, \quad \rho_n = u_n(\frac{1}{2}u_n + u_{n+1})$ $\rho_n = \frac{1}{3}u_n^3 + u_n u_{n+1}(u_n + u_{n+1} + u_{n+2})$
Symmetries	$G = u_x, \quad G = 6uu_x + u_{3x}$ $G = 30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x}$	$G = u_nu_{n+1}\,(u_n + u_{n+1} + u_{n+2}) - u_{n-1}u_n(u_{n-2} + u_{n-1} + u_n)$

Table 2: Prototypical Examples

Differential-difference Equations (DDEs)

- **Given: Nonlinear system of DDEs:**

  (continuous in time, discretized in space)

$$\dot{\mathbf{u}}_n = \mathbf{F}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$$

  where $\mathbf{u}_n$ and $\mathbf{F}$ are vector dynamical variables.

  $\mathbf{F}$ is polynomial with constant coefficients.

  No restrictions on the level of the shifts or the degree of nonlinearity.

- **Conservation Law**:

$$\dot{\rho}_n = J_n - J_{n+1}$$

  with density ρ_n and flux J_n.

  Both are polynomials in $\mathbf{u}_n$ and its shifts.

$$\frac{\mathrm{d}}{\mathrm{dt}}(\sum_n \rho_n) = \sum_n \dot{\rho}_n = \sum_n (J_n - J_{n+1})$$

  if J_n is bounded for all n.

  Subject to suitable boundary or periodicity conditions

$$\sum_n \rho_n = \text{constant}.$$

- **Example.**

  Consider the one-dimensional Toda lattice

$$\ddot{y}_n = \exp(y_{n-1} - y_n) - \exp(y_n - y_{n+1})$$

  y_n is the displacement from equilibrium of the nth particle with unit mass under an exponential decaying interaction force between nearest neighbors.

Change of variables:

$$u_n = \dot{y}_n, \qquad v_n = \exp\left(y_n - y_{n+1}\right)$$

yields

$$\dot{u}_n = v_{n-1} - v_n, \qquad \dot{v}_n = v_n(u_n - u_{n+1}).$$

Toda system is completely integrable.

The first two density-flux pairs (computed by hand):

$$\rho_n^{(1)} = u_n, \quad J_n^{(1)} = v_{n-1}, \quad \text{and} \quad \rho_n^{(2)} = \tfrac{1}{2}u_n^2 + v_n, \quad J_n^{(2)} = u_n v_{n-1}.$$

- **Key concept: Dilation invariance.**

The conservation laws and symmetries are invariant under the dilation symmetry of the Toda lattice:

$$(t, u_n, v_n) \to (\lambda^{-1}t, \lambda u_n, \lambda^2 v_n).$$

Thus, u_n corresponds to one t-derivative: $u_n \sim \frac{\mathrm{d}}{\mathrm{dt}}$.

Similarly, $v_n \sim \frac{\mathrm{d}^2}{\mathrm{dt}^2}$.

Weight, w, of variables are defined in terms of t-derivatives.
With $w(\frac{\mathrm{d}}{\mathrm{dt}}) = 1$, we get $w(u_n) = 1$, $w(v_n) = 2$.

Weights of dependent variables are nonnegative, rational, and independent of n.

The *rank* of a monomial is its total weight in terms of t-derivatives.

- **Equivalence Criterion.**

Define D *shift-down* operator, and U *shift-up* operator,
on the set of all monomials in $\mathbf{u}_n$ and their shifts.

For a monomial m :

$$Dm = m|_{n \to n-1}, \quad \text{and} \quad Um = m|_{n \to n+1}.$$

For example

$$Du_{n+2}v_n = u_{n+1}v_{n-1}, \qquad Uu_{n-2}v_{n-1} = u_{n-1}v_n.$$

Compositions of D and U define an *equivalence relation*.
All shifted monomials are *equivalent*.

For example

$$u_{n-1}v_{n+1} \equiv u_{n+2}v_{n+4} \equiv u_{n-3}v_{n-1}.$$

Equivalence criterion:

Two monomials m_1 and m_2 are equivalent, $m_1 \equiv m_2$, if

$$m_1 = m_2 + [M_n - M_{n+1}]$$

for some polynomial M_n.

For example, $u_{n-2}u_n \equiv u_{n-1}u_{n+1}$ since

$$u_{n-2}u_n = u_{n-1}u_{n+1}+[u_{n-2}u_n-u_{n-1}u_{n+1}] = u_{n-1}u_{n+1}+[M_n-M_{n+1}].$$

Main representative of an equivalence class is the monomial with label n on u (or v).

For example, $u_n u_{n+2}$ is the main representative of the class with elements $u_{n-1}u_{n+1}, u_{n+1}u_{n+3}$, etc.

Use lexicographical ordering to resolve conflicts.

For example, $u_n v_{n+2}$ (not $u_{n-2}v_n$) is the main representative of the class with elements $u_{n-3}v_{n-1}, u_{n+2}v_{n+4}$, etc.

- **Algorithm for Conserved Densities of DDEs.**

Three-step algorithm to find conserved densities:

1). Determine the weights.

2). Construct the form of density.

3). Determine the coefficients.

Example: Density of rank 3 of the Toda lattice,

$$\dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}).$$

Step 1: **Compute the weights.**

Require uniformity in rank for each equation to compute the weights:

$$w(u_n) + w(\frac{\mathrm{d}}{\mathrm{dt}}) = w(v_{n-1}) = w(v_n),$$
$$w(v_n) + w(\frac{\mathrm{d}}{\mathrm{dt}}) = w(v_n) + w(u_n) = w(v_n) + w(u_{n+1})$$

Weights are shift invariant. Set $w(\frac{\mathrm{d}}{\mathrm{dt}}) = 1$ and solve the linear system: $w(u_n) = w(u_{n+1}) = 1$ and $w(v_n) = w(v_{n-1}) = 2$.

Step 2: **Construct the form of the density.**

List all monomials in u_n and v_n of rank 3 or less:

$$\mathcal{G} = \{u_n^3, u_n^2, u_n v_n, u_n, v_n\}.$$

For each monomial in $\mathcal{G}$, introduce enough t-derivatives to obtain weight 3. Use the DDE to remove $\dot{u}_n$ and $\dot{v}_n$:

$$\frac{\mathrm{d}^0}{\mathrm{dt}^0}(u_n^3) = u_n^3, \qquad \frac{\mathrm{d}^0}{\mathrm{dt}^0}(u_n v_n) = u_n v_n,$$
$$\frac{\mathrm{d}}{\mathrm{dt}}(u_n^2) = 2u_n v_{n-1} - 2u_n v_n,$$

$$\frac{\mathrm{d}}{\mathrm{dt}}(v_n) = u_n v_n - u_{n+1} v_n,$$
$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}(u_n) = u_{n-1} v_{n-1} - u_n v_{n-1} - u_n v_n + u_{n+1} v_n.$$

Gather the resulting terms in a set

$$\mathcal{H} = \{u_n^3, u_n v_{n-1}, u_n v_n, u_{n-1} v_{n-1}, u_{n+1} v_n\}.$$

Replace members in the same equivalence class by their main representatives .

For example, $u_n v_{n-1} \equiv u_{n+1} v_n$ are replaced by $u_n v_{n-1}$.

Linearly combine the monomials in

$$\mathcal{I} = \{u_n^3, u_n v_{n-1}, u_n v_n\}$$

to obtain

$$\rho_n = c_1\, u_n^3 + c_2\, u_n v_{n-1} + c_3\, u_n v_n.$$

Step 3: **Determine the coefficients.**

Require that $\dot{\rho}_n = J_n - J_{n+1}$, holds.

Compute $\dot{\rho}_n$ and use the DDE to remove $\dot{u}_n$ and $\dot{v}_n$.

Group the terms

$$\begin{aligned} \dot{\rho}_n &= (3c_1 - c_2) u_n^2 v_{n-1} + (c_3 - 3c_1) u_n^2 v_n + (c_3 - c_2) v_{n-1} v_n \\ &\quad + c_2 u_{n-1} u_n v_{n-1} + c_2 v_{n-1}^2 - c_3 u_n u_{n+1} v_n - c_3 v_n^2. \end{aligned}$$

Use the equivalence criterion to modify $\dot{\rho}_n$.

Replace $u_{n-1} u_n v_{n-1}$ by

$$u_n u_{n+1} v_n + [u_{n-1} u_n v_{n-1} - u_n u_{n+1} v_n].$$

Introduce the main representatives. Thus

$$\begin{aligned}
\dot{\rho}_n &= (3c_1 - c_2)u_n^2 v_{n-1} + (c_3 - 3c_1)u_n^2 v_n \\
&\quad +(c_3 - c_2)v_n v_{n+1} + [(c_3 - c_2)v_{n-1}v_n - (c_3 - c_2)v_n v_{n+1}] \\
&\quad +c_2 u_n u_{n+1} v_n + [c_2 u_{n-1} u_n v_{n-1} - c_2 u_n u_{n+1} v_n] \\
&\quad +c_2 v_n^2 + [c_2 v_{n-1}^2 - c_2 v_n^2] - c_3 u_n u_{n+1} v_n - c_3 v_n^2.
\end{aligned}$$

Group the terms outside of the square brackets and move the pairs inside the square brackets to the bottom.

Rearrange the terms to match the pattern $[J_n - J_{n+1}]$.

Hence

$$\begin{aligned}
\dot{\rho}_n &= (3c_1 - c_2)u_n^2 v_{n-1} + (c_3 - 3c_1)u_n^2 v_n \\
&\quad +(c_3 - c_2)v_n v_{n+1} + (c_2 - c_3)u_n u_{n+1} v_n + (c_2 - c_3)v_n^2 \\
&\quad +[\{(c_3 - c_2)v_{n-1}v_n + c_2 u_{n-1} u_n v_{n-1} + c_2 v_{n-1}^2\} \\
&\quad -\{(c_3 - c_2)v_n v_{n+1} + c_2 u_n u_{n+1} v_n + c_2 v_n^2\}].
\end{aligned}$$

The terms inside the square brackets determine:

$$J_n = (c_3 - c_2)v_{n-1}v_n + c_2 u_{n-1} u_n v_{n-1} + c_2 v_{n-1}^2.$$

The terms outside the square brackets must vanish, thus

$$\mathcal{S} = \{3c_1 - c_2 = 0, c_3 - 3c_1 = 0, c_2 - c_3 = 0\}.$$

The solution is $3c_1 = c_2 = c_3$, so choose $c_1 = \frac{1}{3}$, and $c_2 = c_3 = 1$:

$$\rho_n = \tfrac{1}{3}\, u_n^3 + u_n(v_{n-1} + v_n), \qquad J_n = u_{n-1} u_n v_{n-1} + v_{n-1}^2.$$

Analogously, conserved densities of rank ≤ 5:

$$\rho_n^{(1)} \; = \; u_n \qquad\qquad \rho_n^{(2)} = \tfrac{1}{2}{u_n}^2 + v_n$$

$$\rho_n^{(3)} = \tfrac{1}{3}{u_n}^3 + u_n(v_{n-1} + v_n)$$

$$\rho_n^{(4)} = \tfrac{1}{4}{u_n}^4 + {u_n}^2(v_{n-1} + v_n) + u_n u_{n+1} v_n + \tfrac{1}{2}{v_n}^2 + v_n v_{n+1}$$

$$\begin{aligned}\rho_n^{(5)} &= \tfrac{1}{5}{u_n}^5 + {u_n}^3(v_{n-1} + v_n) + u_n u_{n+1} v_n(u_n + u_{n+1}) \\ &\quad + u_n v_{n-1}(v_{n-2} + v_{n-1} + v_n) + u_n v_n(v_{n-1} + v_n + v_{n+1}).\end{aligned}$$

- **Generalized Symmetries**

A vector function $\mathbf{G}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$ is a *symmetry* if the infinitesimal transformation $\mathbf{u}_n \to \mathbf{u}_n + \epsilon\mathbf{G}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$ leaves the DDE system invariant within order ϵ.
Consequently, $\mathbf{G}$ must satisfy the linearized equation

$$\mathrm{D}_t\mathbf{G} = \mathbf{F}'(\mathbf{u}_n)[\mathbf{G}] = \frac{\partial}{\partial\epsilon}\mathbf{F}(\mathbf{u}_n + \epsilon\mathbf{G})\big|_{\epsilon=0}.$$

where $\mathbf{F}'$ is the Fréchet derivative of $\mathbf{F}$.

Here, $\mathbf{u}_n \to \mathbf{u}_n + \epsilon\mathbf{G}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$ means that $\mathbf{u}_{n+k}$ is replaced by $\mathbf{u}_{n+k} + \epsilon\mathbf{G}_{|n\to n+k}$.

- **Example**

Consider the Toda lattice

$$\dot{u}_n = v_{n-1} - v_n, \qquad \dot{v}_n = v_n(u_n - u_{n+1}).$$

Higher-order symmetry of rank $(3, 4)$:

$$\begin{aligned}G_1 &= v_n(u_n + u_{n+1}) - v_{n-1}(u_{n-1} + u_n), \\ G_2 &= v_n(u_{n+1}^2 - u_n^2) + v_n(v_{n+1} - v_{n-1}).\end{aligned}$$

- **Algorithm for Generalized Symmetries of DDEs.**

Consider the Toda system with $w(u_n) = 1$ and $w(v_n) = 2$.
Compute the form of the symmetry of ranks $(3, 4)$, i.e. the first component of the symmetry has rank 3, the second rank 4.

Step 1: **Construct the form of the symmetry.**

List all monomials in u_n and v_n of rank 3 or less:

$$\mathcal{L}_1 = \{u_n^3, u_n^2, u_n v_n, u_n, v_n\},$$

and of rank 4 or less:

$$\mathcal{L}_2 = \{u_n^4, u_n^3, u_n^2 v_n, u_n^2, u_n v_n, u_n, v_n^2, v_n\}.$$

For each monomial in $\mathcal{L}_1$ and $\mathcal{L}_2$, introduce enough t-derivatives, so that each term exactly has rank 3 and 4, respectively.

Using the DDEs, for the monomials in $\mathcal{L}_1$:

$$\begin{aligned}
&\frac{\mathrm{d}^0}{\mathrm{dt}^0}(u_n^3) = u_n^3, \qquad \frac{\mathrm{d}^0}{\mathrm{dt}^0}(u_n v_n) = u_n v_n, \\
&\frac{\mathrm{d}}{\mathrm{dt}}(u_n^2) = 2u_n \dot{u}_n = 2u_n v_{n-1} - 2u_n v_n, \\
&\frac{\mathrm{d}}{\mathrm{dt}}(v_n) = \dot{v}_n = u_n v_n - u_{n+1} v_n, \\
&\frac{\mathrm{d}^2}{\mathrm{dt}^2}(u_n) = \frac{\mathrm{d}}{\mathrm{dt}}(\dot{u}_n) = \frac{\mathrm{d}}{\mathrm{dt}}(v_{n-1} - v_n) \\
&\quad = u_{n-1} v_{n-1} - u_n v_{n-1} - u_n v_n + u_{n+1} v_n.
\end{aligned}$$

Gather the resulting terms:

$$\mathcal{R}_1 = \{u_n^3, u_{n-1} v_{n-1}, u_n v_{n-1}, u_n v_n, u_{n+1} v_n\}.$$

$$
\begin{aligned}
\mathcal{R}_2 \;=\; & \{u_n^4, u_{n-1}^2 v_{n-1}, u_{n-1}u_n v_{n-1}, u_n^2 v_{n-1}, v_{n-2}v_{n-1}, v_{n-1}^2, u_n^2 v_n, \\
& u_n u_{n+1} v_n, u_{n+1}^2 v_n, v_{n-1}v_n, v_n^2, v_n v_{n+1}\}.
\end{aligned}
$$

Linearly combine the monomials in $\mathcal{R}_1$ and $\mathcal{R}_2$

$$
\begin{aligned}
G_1 &= c_1\, u_n^3 + c_2\, u_{n-1}v_{n-1} + c_3\, u_n v_{n-1} + c_4\, u_n v_n + c_5\, u_{n+1}v_n, \\
G_2 &= c_6\, u_n^4 + c_7\, u_{n-1}^2 v_{n-1} + c_8\, u_{n-1}u_n v_{n-1} + c_9\, u_n^2 v_{n-1} \\
&\quad + c_{10}\, v_{n-2}v_{n-1} + c_{11}\, v_{n-1}^2 + c_{12}\, u_n^2 v_n + c_{13}\, u_n u_{n+1} v_n \\
&\quad + c_{14}\, u_{n+1}^2 v_n + c_{15}\, v_{n-1}v_n + c_{16}\, v_n^2 + c_{17}\, v_n v_{n+1}.
\end{aligned}
$$

Step 2: Determine the unknown coefficients.

Require that the symmetry condition holds.

Solution:

$$
\begin{gathered}
c_1 = c_6 = c_7 = c_8 = c_9 = c_{10} = c_{11} = c_{13} = c_{16} = 0, \\
-c_2 = -c_3 = c_4 = c_5 = -c_{12} = c_{14} = -c_{15} = c_{17}.
\end{gathered}
$$

Therefore, with $c_{17} = 1$, the symmetry of rank $(3, 4)$ is:

$$
\begin{aligned}
G_1 &= u_n v_n - u_{n-1}v_{n-1} + u_{n+1}v_n - u_n v_{n-1}, \\
G_2 &= u_{n+1}^2 v_n - u_n^2 v_n + v_n v_{n+1} - v_{n-1}v_n.
\end{aligned}
$$

Analogously, the symmetry of rank $(4, 5)$ reads

$$
\begin{aligned}
G_1 &= u_n^2 v_n + u_n u_{n+1} v_n + u_{n+1}^2 v_n + v_n^2 + v_n v_{n+1} - u_{n-1}^2 v_{n-1} \\
&\quad - u_{n-1}u_n v_{n-1} - u_n^2 v_{n-1} - v_{n-2}v_{n-1} - v_{n-1}^2, \\
G_2 &= u_{n+1}v_n^2 + 2u_{n+1}v_n v_{n+1} + u_{n+2}v_n v_{n+1} - u_n^3 v_n + u_{n+1}^3 v_n \\
&\quad - u_{n-1}v_{n-1}v_n - 2u_n v_{n-1}v_n - u_n v_n^2.
\end{aligned}
$$

- **Example: The Ablowitz-Ladik DDE.**

  Consider the Ablowitz and Ladik discretization,

$$i\,\dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + \kappa u_n^* u_n (u_{n+1} + u_{n-1}),$$

  of the NLS equation,

$$iu_t + u_{xx} + \kappa u^2 u^* = 0$$

  u_n^* is the complex conjugate of u_n. Treat u_n and $v_n = u_n^*$ as independent variables and add the complex conjugate equation. Set $\kappa = 1$ (scaling) and absorb i in the scale on t :

$$\begin{aligned} \dot{u}_n &= u_{n+1} - 2u_n + u_{n-1} + u_n v_n (u_{n+1} + u_{n-1}), \\ \dot{v}_n &= -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}). \end{aligned}$$

  Since $v_n = u_n^*$, $w(v_n) = w(u_n)$.

  No uniformity in rank! Introduce an auxiliary parameter α with weight.

$$\begin{aligned} \dot{u}_n &= \alpha(u_{n+1} - 2u_n + u_{n-1}) + u_n v_n (u_{n+1} + u_{n-1}), \\ \dot{v}_n &= -\alpha(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}). \end{aligned}$$

  Uniformity in rank leads to

$$\begin{aligned} w(u_n) + 1 &= w(\alpha) + w(u_n) = 2w(u_n) + w(v_n) = 3w(u_n), \\ w(v_n) + 1 &= w(\alpha) + w(v_n) = 2w(v_n) + w(u_n) = 3w(v_n). \end{aligned}$$

  which yields

$$w(u_n) = w(v_n) = \frac{1}{2}, \quad w(\alpha) = 1.$$

Conserved densities (for $\alpha = 1$, in original variables):

$$\rho_n^{(1)} = u_n u_{n-1}^*$$

$$\rho_n^{(2)} = u_n u_{n+1}^*$$

$$\rho_n^{(3)} = \tfrac{1}{2} u_n^2 u_{n-1}^{*2} + u_n u_{n+1} u_{n-1}^* v_n + u_n u_{n-2}^*$$

$$\rho_n^{(4)} = \tfrac{1}{2} u_n^2 u_{n+1}^{*2} + u_n u_{n+1} u_{n+1}^* u_{n+2}^* + u_n u_{n+2}^*$$

$$\begin{aligned}\rho_n^{(5)} = {} & \tfrac{1}{3} u_n^3 u_{n-1}^{*3} + u_n u_{n+1} u_{n-1}^* u_n^* (u_n u_{n-1}^* + u_{n+1} u_n^* + u_{n+2} u_{n+1}^*) \\ & + u_n u_{n-1}^* (u_n u_{n-2}^* + u_{n+1} u_{n-1}^*) + u_n u_n^* (u_{n+1} u_{n-2}^* + u_{n+2} u_{n-1}^*) + u_n u_{n-3}^*\end{aligned}$$

$$\begin{aligned}\rho_n^{(6)} = {} & \tfrac{1}{3} u_n^3 u_{n+1}^{*3} + u_n u_{n+1} u_{n+1}^* u_{n+2}^* (u_n u_{n+1}^* + u_{n+1} u_{n+2}^* + u_{n+2} u_{n+3}^*) \\ & + u_n u_{n+2}^* (u_n u_{n+1}^* + u_{n+1} u_{n+2}^*) + u_n u_{n+3}^* (u_{n+1} u_{n+1}^* + u_{n+2} u_{n+2}^*) + u_n u_{n+3}^*\end{aligned}$$

The Ablowitz-Ladik lattice has infinitely many conserved densities.

Although a constant of motion, we cannot find the Hamiltonian:

$$H = -i \sum [u_n^* (u_{n-1} + u_{n+1}) - 2 \ln(1 + u_n u_n^*)],$$

since it has a logarithmic term.

- **Application: Discretization of the combined KdV-mKdV equation.**

Consider the integrable discretization

$$\begin{aligned}\dot{u}_n &= -(1+\alpha h^2 u_n + \beta h^2 u_n^2)\,\Big\{\, \tfrac{1}{h^3}\,(\tfrac{1}{2}u_{n+2} - u_{n+1} + u_{n-1} - \tfrac{1}{2}u_{n-2}) \\ &\quad + \tfrac{\alpha}{2h}[u_{n+1}^2 - u_{n-1}^2 + u_n(u_{n+1}-u_{n-1}) + u_{n+1}u_{n+2} - u_{n-1}u_{n-2}] \\ &\quad + \tfrac{\beta}{2h}[u_{n+1}^2(u_{n+2}+u_n) - u_{n-1}^2(u_{n-2}+u_n)]\,\Big\}\end{aligned}$$

of the combined KdV-mKdV equation

$$u_t + 6\alpha u u_x + 6\beta u^2 u_x + u_{xxx} = 0.$$

Discretizations the KdV and mKdV equations are special cases.

Set $h = 1$ (scaling). No uniformity in rank!

Introduce auxiliary parameters γ and δ with weights.

$$\begin{aligned}\dot{u}_n &= -(\gamma + \alpha u_n + \beta u_n^2)\,\Big\{\, \delta(\tfrac{1}{2}u_{n+2} - u_{n+1} + u_{n-1} - \tfrac{1}{2}u_{n-2}) \\ &\quad + \tfrac{\alpha}{2}[u_{n+1}^2 - u_{n-1}^2 + u_n(u_{n+1}-u_{n-1}) + u_{n+1}u_{n+2} - u_{n-1}u_{n-2}] \\ &\quad + \tfrac{\beta}{2}[u_{n+1}^2(u_{n+2}+u_n) - u_{n-1}^2(u_{n-2}+u_n)]\,\Big\},\end{aligned}$$

Uniformitiy in rank requires

$$w(\gamma) = w(\delta) = 2w(u_n), \quad w(\alpha) = w(u_n), \quad w(\beta) = 0.$$

Then,

$$w(u_n) + 1 = 5w(u_n),$$

Hence,

$$w(u_n) = w(\alpha) = \tfrac{1}{4}, \quad w(\gamma) = w(\delta) = \tfrac{1}{2}, \quad w(\beta) = 0,$$

Conserved densities: Special cases

For the KdV case ($\beta = 0$) :

$$\dot{u}_n \;=\; -(\gamma + \alpha h^2 u_n) \left\{ \tfrac{\delta}{h^3}(\tfrac{1}{2}u_{n+2} - u_{n+1} + u_{n-1} - \tfrac{1}{2}u_{n-2}) \right. \\ \left. + \tfrac{\alpha}{2h}[u_{n+1}^2 - u_{n-1}^2 + u_n(u_{n+1} - u_{n-1}) + u_{n+1}u_{n+2} - u_{n-1}u_{n-2}] \right\}$$

with $\gamma = \delta = 1$ is a completely integrable discretization of the KdV equation

$$u_t + 6\alpha u u_x + u_{xxx} = 0.$$

Now,

$$w(\gamma) = w(\delta) = w(u_n), \;\; w(\alpha) = 0.$$

Then,

$$w(u_n) + 1 = 3w(u_n).$$

So,

$$w(u_n) = w(\gamma) = w(\delta) = \tfrac{1}{2}, \quad w(\alpha) = 0.$$

From rank $\frac{3}{2}$ and $\frac{5}{2}$ (after splitting):

$$\rho_n^{(1)} \;=\; u_n,$$

$$\rho_n^{(2)} \;=\; u_n(\tfrac{1}{2}u_n + u_{n+1}),$$

$$\rho_n^{(3)} \;=\; u_n(\tfrac{1}{3}u_n^2 + u_n u_{n+1} + u_{n+1}^2 + \tfrac{1}{\alpha}u_{n+2} + u_{n+1}u_{n+2})$$

$$\rho_n^{(4)} \;=\; u_n(\tfrac{1}{4}u_n^3 + u_n^2 u_{n+1} + \tfrac{3}{2}u_n u_{n+1}^2 + u_{n+1}^3 + \cdots + u_{n+1}u_{n+2}u_{n+3})$$

$$\rho_n^{(5)} \;=\; u_n(\tfrac{1}{5}\alpha u_n^4 - \tfrac{1}{2}u_n^3 - 2u_n^2 u_{n+1} + \cdots + \alpha u_{n+1}u_{n+2}u_{n+3}u_{n+4})$$

For the mKdV case ($\alpha = 0$) :

$$\begin{aligned} \dot{u}_n &= -(\gamma + \beta h^2 u_n^2) \left\{ \tfrac{\delta}{h^3}(\tfrac{1}{2}u_{n+2} - u_{n+1} + u_{n-1} - \tfrac{1}{2}u_{n-2}) \right. \\ & \left. \quad + \tfrac{\beta}{2h}[u_{n+1}^2(u_{n+2} + u_n) - u_{n-1}^2(u_{n-2} + u_n)] \right\} \end{aligned}$$

with $\gamma = \delta = 1$ is a completely integrable discretization of the modified KdV equation

$$u_t + 6\beta u^2 u_x + u_{xxx} = 0.$$

Now,

$$w(\gamma) = w(\delta) = 2w(u_n), \quad w(\beta) = 0.$$

Then,

$$w(u_n) + 1 = 5w(u_n).$$

So,

$$w(u_n) = \tfrac{1}{4}, \quad w(\gamma) = w(\delta) = \tfrac{1}{2}, \quad w(\beta) = 0.$$

From rank $\frac{3}{2}$ and $\frac{5}{2}$ (after splitting):

$$\begin{aligned} \rho_n^{(1)} &= u_n u_{n+1}, \\ \rho_n^{(2)} &= u_n(\tfrac{1}{2}u_n u_{n+1}^2 + \tfrac{1}{\beta}u_{n+2} + u_{n+1}^2 u_{n+2}) \\ \rho_n^{(3)} &= u_n(\tfrac{1}{3}u_n^2 u_{n+1}^3 + \tfrac{1}{\beta}u_n u_{n+1} u_{n+2} + \cdots + u_{n+1}^2 u_{n+2}^2 u_{n+3}) \\ \rho_n^{(4)} &= u_n(\tfrac{1}{4}\beta u_n^3 u_{n+1}^4 + u_n^2 u_{n+1}^2 u_{n+2} + \cdots + \beta u_{n+1}^2 u_{n+2}^2 u_{n+3}^2 u_{n+4}) \end{aligned}$$

Mathematica Software

- **Scope and Limitations of Algorithms.**
  - Systems of PDEs and DDEs must be polynomial in dependent variables.
    No *explicitly* dependencies on the independent variables.
  - Only one space variable (continuous or discretized) is allowed.
  - Program only computes polynomial conservation laws and generalized symmetries (no recursion operators yet).
  - Program computes conservation laws and symmetries that explicitly depend on the independent variables, if the highest degree is specified.
  - No limit on the number of equations in the system.
    In practice: time and memory constraints.
  - Input systems may have (nonzero) parameters.
    Program computes the compatibility conditions for parameters such that conservation laws and symmetries (of a given rank) exist.
  - Systems can also have parameters with (unknown) weight.
    This allows one to test evolution and lattice equations of non-uniform rank.
  - For systems where one or more of the weights is free, the program prompts the user for info.
  - Fractional weights and ranks are permitted.
  - Complex dependent variables are allowed.
  - PDEs and lattice equations must be of first-order in t.

- **Conclusions and Future Research**
  - Implement the recursion operator algorithm for PDEs.
  - Design an algorithm for recursion operators of DDEs.
  - Add tools for parameter analysis (Gröbner basis).
  - Exploit other symmetries in the hope to find conserved densities. of non-polynomial form
  - Application: test model DDEs for integrability.

    (study the integrable discretization of KdV-mKdV equation).
  - Compute constants of motion for dynamical systems (e.g. Lorenz and Hénon-Heiles systems)

- **Implementation in Mathematica – Software**
  - Ü. Göktaş and W. Hereman, The software package *InvariantsSymmetries.m* and the related files are available at http://www.mathsource.com/cgi-bin/msitem?0208-932. *MathSource* is an electronic library of *Mathematica* material.
  - Software: available via FTP, ftp site *mines.edu* in

    pub/papers/math_cs_dept/software/condens
    pub/papers/math_cs_dept/software/diffdens

    or via the Internet

    URL: http://www.mines.edu/fs_home/whereman/

- **Publications**

1). Ü. Göktaş and W. Hereman, Symbolic computation of conserved densities for systems of nonlinear evolution equations, J. Symbolic Computation, 24 (1997) 591–621.

2). Ü. Göktaş, W. Hereman, and G. Erdmann, Computation of conserved densities for systems of nonlinear differential-difference equations, Phys. Lett. A, 236 (1997) 30–38.

3). Ü. Göktaş and W. Hereman, Computation of conserved densities for nonlinear lattices, Physica D, 123 (1998) 425–436.

4). Ü. Göktaş and W. Hereman, Algorithmic computation of higher-order symmetries for nonlinear evolution and lattice equations, Advances in Computational Mathematics 11 (1999), 55-80.

5). W. Hereman and Ü. Göktaş, Integrability Tests for Nonlinear Evolution Equations. In: *Computer Algebra Systems: A Practical Guide*, Ed.: M. Wester, Wiley and Sons, New York (1999) Chapter 12, pp. 211-232.

6). W. Hereman, Ü. Göktaş, M. Colagrosso, and A. Miller, Algorithmic integrability tests for nonlinear differential and lattice equations, Computer Physics Communications 115 (1998) 428–446.