SOLITARY WAVE SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS USING A DIRECT METHOD AND MACSYMA

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1. INTRODUCTION

- Goal: Exact solutions

    Single solitary wave or soliton solutions
    N-solitons
    Implicit solutions

- Applicable to:

    Single nonlinear evolution and wave equations
    Systems of nonlinear PDEs
    Nonlinear ODEs

- Method:

    Hirota's direct method
    Rosales' perturbation method
    Trace method
    Hereman *et al* real exponential approach
    Frobenius method

- Requirements :

    Based on physical principles
    Simple and straightforward
    Programmable in MACSYMA, REDUCE,
    MATHEMATICA, SCRATCHPAD II, etc.

2. EXAMPLES

- Korteweg-de Vries equation and generalizations

$$u_t + au^n u_x + u_{xxx} = 0, \quad n \in \mathbb{N}$$

$$u(x,t) = \left\{ \frac{c(n+1)(n+2)}{2a} \operatorname{sech}^2 \left[\frac{n}{2} \sqrt{c}(x - ct) + \delta \right] \right\}^{\frac{1}{n}}$$

- Burgers equation

$$u_t + auu_x - u_{xx} = 0$$

$$u(x,t) = \frac{c}{a} \left\{ 1 - \tanh \left[\frac{c}{2}(x - ct) + \delta \right] \right\}$$

- Fisher equation and generalizations

$$u_t - u_{xx} - u(1 - u^n) = 0, \quad n \in \mathbb{N}$$

$$u(x,t) = \left\{ \frac{1}{2} \left[1 - \tanh \left[\frac{n}{2\sqrt{2n+4}} (x - \frac{(n+4)}{\sqrt{2n+4}} t) + \delta \right] \right] \right\}^{\frac{2}{n}}$$

- Fitzhugh-Nagumo equation

$$u_t - u_{xx} + u(1-u)(a-u) = 0$$

$$u(x,t) = \frac{a}{2}\left\{1 + \tanh\left[\frac{a}{2\sqrt{2}}(x - \frac{(2-a)}{\sqrt{2}}t) + \delta\right]\right\}$$

- Kuramoto-Sivashinski equation

$$u_t + uu_x + au_{xx} + bu_{xxxx} = 0$$

$$\begin{aligned} u(x,t) &= c + \frac{165ak}{19}\left\{\tanh^3\left[\frac{k(x-ct)}{2} + \delta\right]\right\} \\ &- \frac{135ak}{19}\left\{\tanh\left[\frac{k(x-ct)}{2} + \delta\right]\right\} \end{aligned}$$

with $k = \sqrt{\frac{11a}{19b}}$

$$\begin{aligned} u(x,t) &= c - \frac{15ak}{19}\left\{\tanh^3\left[\frac{k(x-ct)}{2} + \delta\right]\right\} \\ &+ \frac{45ak}{19}\left\{\tanh\left[\frac{k(x-ct)}{2} + \delta\right]\right\} \end{aligned}$$

with $k = \sqrt{\frac{-a}{19b}}$

- Harry Dym equation

$$u_t + (1-u)^3 u_{xxx} = 0$$

$$\begin{aligned} u(x,t) &= \operatorname{sech}^2\left[\frac{1}{2}\sqrt{c}\,[x - ct + \delta(x,t)]\right] \\ \delta(x,t) &= \frac{2}{\sqrt{c}}\tanh\left[\frac{\sqrt{c}}{2}\,[x - ct + \delta(x,t)]\right] \end{aligned}$$

- sine-Gordon equation

$$u_{tt} - u_{xx} - \sin u = 0$$

$$u(x,t) = 4\arctan\left\{\exp\left[\frac{1}{\sqrt{-c}}(x - ct) + \delta\right]\right\}$$

- Coupled Korteweg-de Vries equations

$$u_t - a(6uu_x + u_{xxx}) - 2b\, vv_x = 0,$$
$$v_t + 3uv_x + v_{xxx} = 0$$

$$\begin{aligned} u(x,t) &= 2\,c\,\mathrm{sech}^2\left[\sqrt{c}(x-ct)+\delta\right], \\ v(x,t) &= \pm c\sqrt{\frac{-2(4a+1)}{b}}\,\mathrm{sech}\left[\sqrt{c}(x-ct)+\delta\right], \end{aligned}$$

$$\begin{aligned} u(x,t) &= c\,\mathrm{sech}^2\left[\frac{1}{2}\sqrt{c}(x-ct)+\delta\right] \\ v(x,t) &= \frac{3}{\sqrt{6|b|}}\,u(x,t) = \frac{3\,c}{\sqrt{6|b|}}\,\mathrm{sech}^2\left[\frac{1}{2}\sqrt{c}(x-ct)+\delta\right] \end{aligned}$$

3. THE ALGORITHM

- Step 1: System of two coupled nonlinear PDEs

$$\begin{aligned}\mathcal{F}(u, v, u_t, u_x, v_t, v_x, u_{tx}, ..., u_{mx}, v_{nx}) &= 0, \\ \mathcal{G}(u, v, u_t, u_x, v_t, v_x, u_{tx}, ..., u_{px}, v_{qx}) &= 0, \quad (m, n, p, q \in \mathbb{N})\end{aligned}$$

  where $\mathcal{F}$ and $\mathcal{G}$ are polynomials in their arguments and

$$u_t = \frac{\partial u}{\partial t}, \quad u_{nx} = \frac{\partial^n u}{\partial x^n}$$

- Step 2:
  - Introduce the variable $\xi = x - ct$, (c is the constant velocity)
  - Integrate the system of ODEs for $\phi(\xi) \equiv u(x,t)$ and $\psi(\xi) \equiv v(x,t)$ with respect to ξ to reduce the order
  - Ignore integration constants and assume that ϕ and ψ and their derivatives vanish at $\xi = \pm\infty$
  - Carry out a nonlinear transformation

$$\phi = \tilde{\phi}^{\alpha}, \quad \psi = \tilde{\psi}^{\beta}$$

- Step 3:
  - Expand $\tilde{\phi}$ and $\tilde{\psi}$ in a power series

    $$\phi = \sum_{n=1}^{\infty} a_n\, g^n, \quad \psi = \sum_{n=1}^{\infty} b_n\, g^n$$
  - $g(\xi) = \exp[-K(c)\xi]$ solves the linear part of at least one of the equations
  - Consider the dispersion laws $K(c)$ of (one of) the linearized equations
  - Substitute the expansions into the full nonlinear system
  - Use Cauchy's rule for multiple series to rearrange the multiple sums
  - Equate the coefficient of g^n to get the coupled recursion relations for a_n and b_n
- Step 4:
  - Assume that a_n and b_n are polynomials in n
  - Determine their degrees δ_1 and δ_2
  - Substitute

    $$a_n = \sum_{j=0}^{\delta_1} A_j\, n^j, \quad b_n = \sum_{j=0}^{\delta_2} B_j\, n^j$$

    into the recursion relations

  - Compute the sums by using the formulae for

$$S_k = \sum_{i=1}^{n} i^k, \quad (k = 0, 1, 2, ...)$$

  - Examples:

$$S_0 = n, \quad S_1 = \frac{n(n+1)}{2},$$

$$S_2 = \frac{n(n+1)(2n+1)}{6}, \quad \text{etc.}$$

  - Equate to zero the different coefficients of the polynomial in n
  - Solve the algebraic (nonlinear) equations for the constant coefficients A_j and B_j

- Step 5:
  - Find the closed forms for

$$\begin{aligned} \tilde{\phi} &= \sum_{n=1}^{\infty} \sum_{j=0}^{\delta_1} A_j \, n^j \, g^n \equiv \sum_{j=0}^{\delta_1} A_j \, F_j(g), \\ \tilde{\psi} &= \sum_{n=1}^{\infty} \sum_{j=0}^{\delta_2} B_j \, n^j \, g^n \equiv \sum_{j=0}^{\delta_2} B_j \, F_j(g) \end{aligned}$$

    with

$$\begin{aligned} F_j(g) &\equiv \sum_{n=1}^{\infty} n^j g^n, \\ F_{j+1}(g) &= g F_j'(g), \quad (j = 0, 1, 2, ...) \end{aligned}$$

– Examples:

$$F_0(g) = \frac{g}{1-g}, \quad F_1(g) = \frac{g}{(1-g)^2},$$

$$F_2(g) = \frac{g(1+g)}{(1-g)^3}, \quad \text{etc.}$$

– Return to ϕ and ψ and then to the original variables x and t to obtain the travelling wave solution(s) $u(x,t)$ and $v(x,t)$

4. EXAMPLE: A class of generalized KdV equations

- Step 1:
  - Consider the gKdV equation

$$u_t + (a + bu^q)u^q u_x + u_{3x} = 0$$

with $a, b \in \mathbb{R}; q \in \mathbb{Q}$

- Step 2:
  - Introduce the variable $\xi = x - ct$, c is the constant velocity
  - Integrate the equation for $\phi(\xi) \equiv u(x,t)$ w.r.t. ξ

$$-c\phi + (\frac{a}{q+1} + \frac{b}{2q+1}\phi^q)\phi^{q+1} + \phi_{2\xi} = 0$$

  - Use ϕ_ξ as an integrating factor and integrate again

$$-\frac{c\phi^2}{2} + [\frac{a}{(q+1)(q+2)} + \frac{b}{2(2q+1)(q+1)}\phi^q]\phi^{q+2} + \frac{\phi_\xi^2}{2} = 0$$

- Step 3:
  - Singularity analysis determines the transformation
$$\phi = \tilde{\phi}^{\frac{1}{q}}$$
  - Clear the denominators
$$-\frac{c\tilde{\phi}^2}{2} + [\frac{a}{(q+1)(q+2)} + \frac{b}{2(2q+1)(q+1)}\tilde{\phi}]\tilde{\phi}^3 + \frac{\tilde{\phi}_\xi^2}{2q^2} = 0$$
  - Expand $\tilde{\phi}$ in a power series
$$\tilde{\phi} = \sum_{n=1}^{\infty} a_n \, g^n$$
  - $g(\xi) = \exp[-K(c)\xi]$ solves the linear part of the equation
  - Consider the dispersion law $K(c) = q\sqrt{c}$
  - Substitute the expansion into the full nonlinear equation
  - Use Cauchy's rule for multiple series to rearrange the sums

- Equate the coefficient of g^n to get the nonlinear recursion relation for $n \geq 4$

$$\begin{aligned} c \ & \sum_{l=1}^{n-1} [(n-l)l - 1] a_{n-l} a_l \\ + \ & \frac{2a}{(q+1)(q+2)} \sum_{l=2}^{n-1} \sum_{m=1}^{l-1} a_{n-l} a_{l-m} a_m \\ + \ & \frac{b}{(2q+1)(q+1)} \sum_{l=3}^{n-1} \sum_{m=2}^{l-1} \sum_{j=1}^{m-1} a_{n-l} a_{l-m} a_{m-j} a_j = 0 \end{aligned}$$

- Step 4:
  - Assume that a_n is a polynomial in n
  - Determine the degree $\longrightarrow$ two cases:
- CASE 1: $a \neq 0, b = 0$
  - Substitute $a_n = A_1 n + A_0$ into the recursion relation
  - Compute the sums by using the formulae for

$$S_k = \sum_{i=1}^{n} i^k \quad (k = 0, 1, ..., 5)$$

  - Equate to zero the coefficients of the polynomial in n (of degree 6)

  - Solve the algebraic (nonlinear) equations for the constant coefficients A_1 and A_0
  - Solution calculated with MACSYMA:

$$A_1 = -\frac{2c(q+2)(q+1)}{a}, \quad A_0 = 0$$

- CASE 2: $b \neq 0$
  - Substitute $a_n = A_0$
  - Proceed as in Case 1
  - Solution of the nonlinear system

$$A_1 = 0, \quad A_0 = \frac{a(2q+1)}{b(q+2)}, \quad c = -\frac{a^2(2q+1)}{b(q+1)(q+2)^2}$$

- Step 5:
  - Find the closed form for $\tilde{\phi}$
  - Use $F_0(g) = \dfrac{g}{1-g}$ and $F_1(g) = \dfrac{g}{(1-g)^2}$

$$\begin{aligned}
\tilde{\phi} &= \sum_{n=1}^{\infty}(A_1 n + A_0)a_0 g^n \\
&= [A_1 F_1(a_0 g) + A_0 F_0(a_0 g)] \\
&= [\frac{A_1 a_0 g}{(1-a_0 g)^2} + \frac{A_0 a_0 g}{(1-a_0 g)}] \\
&= -[\frac{A_1 \mathrm{exp}(-K\xi - \Delta)}{(1+\mathrm{exp}(-K\xi-\Delta))^2} + \frac{A_0 \mathrm{exp}(-K\xi - \Delta)}{(1+\mathrm{exp}(-K\xi-\Delta))}] \\
&= -[\frac{A_1}{4}\mathrm{sech}^2(\frac{K\xi+\Delta}{2}) + \frac{A_0}{2}(1-\tanh(\frac{K\xi+\Delta}{2}))]
\end{aligned}$$

  where $a_0 = -\mathrm{exp}(-\Delta)$
  - Return to the original variables x and t

- CASE 1: $a \neq 0, b = 0$:

  $$u(x,t) = \phi(x-ct) = \{\frac{c(q+2)(q+1)}{2a}\mathrm{sech}^2[\frac{q}{2}\sqrt{c}(x-ct)+\frac{\Delta}{2}]\}^{\frac{1}{n}}$$

  with arbitrary velocity c

- CASE 2: $b \neq 0$:

  $$u(x,t) = \phi(x-ct) = \{\frac{-a(2q+1)}{2b(q+2)}(1-\tanh[\frac{q}{2}\sqrt{c}(x-ct)+\frac{\Delta}{2}])\}^{\frac{1}{n}}$$

  with $c = -\frac{a^2(2q+1)}{b(q+1)(q+2)^2}$

Special Cases

Equation	Solution
$u_t + 6uu_x + u_{3x} = 0$	$2k^2\text{sech}^2[k(x - 4k^2t) + \delta]$
$u_t + 6u^2u_x + u_{3x} = 0$	$k^2\text{sech}[k(x - k^2t) + \delta]$
$u_t + (a + bu)uu_x + u_{3x} = 0$	$-(\frac{a}{2b})\{1 - \tanh[\frac{a}{12}\sqrt{\frac{-6}{b}}(x + \frac{a^2}{6b}t) + \delta]\}$
$u_t + (a + bu^2)u^2u_x + u_{3x} = 0$	$\frac{1}{2}\sqrt{\frac{-5a}{2b}}\{1 - \tanh[\frac{a}{12}\sqrt{\frac{-15}{b}}(x + \frac{5a^2}{48b}t) + \delta]\}$
$u_t + 10u^3u_x + u_{3x} = 0$	$k^{\frac{2}{3}}\text{sech}^{\frac{2}{3}}[\frac{3k}{2}(x - k^2t) + \delta]$
$u_t + \sqrt{u}u_x + u_{3x} = 0$	$900k^4\text{sech}^4[k(x - 16k^2t) + \delta]$
$u_t + (a + b\sqrt{u})\sqrt{u}u_x + u_{3x} = 0$	$\frac{4a^2}{25b^2}\{1 - \tanh[\frac{a}{15}\sqrt{\frac{-3}{b}}(x + \frac{16a^2}{75b}t) + \delta]\}^2$