Solving Nonlinear Wave Equations and Lattices with Mathematica

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Purpose & Motivation

- Develop various symbolic algorithms to compute exact solutions of nonlinear (systems) of partial differential equations (PDEs) and differential-difference equations (DDEs, lattices).
- Fully **automate** the tanh and sech methods to compute closed form solitary wave solutions.
- Solutions of tanh or sech type **model** solitary waves in fluid dynamics, plasmas, electrical circuits, optical fibers, bio-genetics, etc.
- Class of nonlinear PDEs and DDEs solvable with the tanh/sech method includes famous evolution and wave equations.
  Typical examples: Korteweg-de Vries, Fisher and Boussinesq PDEs, Toda and Volterra lattices (DDEs).
- **Research aspect**: Design a high-quality application package for the computation of exact solitary wave solutions of large classes of nonlinear evolution and wave equations.
- **Educational aspect**: Software as course ware for courses in nonlinear PDEs, theory of nonlinear waves, integrability, dynamical systems, and modeling with symbolic software.
- Users: scientists working on nonlinear wave phenomena in fluid dynamics, nonlinear networks, elastic media, chemical kinetics, material science, bio-sciences, plasma physics, and nonlinear optics.

Typical Examples of Single PDEs and Systems of PDEs

- The Korteweg-de Vries (KdV) equation:

$$u_t + \alpha u u_x + u_{3x} = 0.$$

Solitary wave solution:

$$u(x,t) = \frac{8c_1^3 - c_2}{6\alpha c_1} - \frac{2c_1^2}{\alpha} \tanh^2 \left[\mathrm{c_1 x + c_2 t} + \delta\right],$$

or, equivalently,

$$u(x,t) = -\frac{4c_1^3 + c_2}{6\alpha c_1} + \frac{2c_1^2}{\alpha} \operatorname{sech}^2 \left[\mathrm{c_1 x + c_2 t} + \delta\right].$$

- The modified Korteweg-de Vries (mKdV) equation:

$$u_t + \alpha u^2 u_x + u_{3x} = 0.$$

Solitary wave solution:

$$u(x,t) = \pm\sqrt{\frac{6}{\alpha}}\, c_1 \ \operatorname{sech}\left[\mathrm{c_1 x - c_1^3 t} + \delta\right].$$

- The Fisher equation:

$$u_t - u_{xx} - u\,(1-u) = 0.$$

Solitary wave solution:

$$u(x,t) = \frac{1}{4} \pm \frac{1}{2} \tanh\xi + \frac{1}{4} \tanh^2\xi,$$

with

$$\xi = \pm\frac{1}{2\sqrt{6}}\, x \pm \frac{5}{12}\, t + \delta.$$

- The generalized Kuramoto-Sivashinski equation:

$$u_t + uu_x + u_{xx} + \sigma u_{3x} + u_{4x} = 0.$$

Solitary wave solutions

(ignoring symmetry $u \to -u, x \to -x, \sigma \to -\sigma$) :

For $\sigma = 4$:

$$u(x,t) = 9 - 2c_2 - 15 \text{ tanh}\xi \ (1 + \text{ tanh}\xi - \text{ tanh}^2\xi),$$

with $\xi = \frac{x}{2} + c_2 t + \delta$.

For $\sigma = 0$:

$$u(x,t) = -2\sqrt{\frac{19}{11}c_2} - \frac{135}{19}\sqrt{\frac{11}{19}} \text{ tanh}\xi + \frac{165}{19}\sqrt{\frac{11}{19}} \text{ tanh}^3\xi,$$

with $\xi = \frac{1}{2}\sqrt{\frac{11}{19}}\ x + c_2 t + \delta$.

For $\sigma = \frac{12}{\sqrt{47}}$:

$$\begin{aligned} u(x,t) &= \frac{45 \mp 4418 c_2}{47\sqrt{47}} \pm \frac{45}{47\sqrt{47}} \text{ tanh}\xi \\ &- \frac{45}{47\sqrt{47}} \text{ tanh}^2\xi \pm \frac{15}{47\sqrt{47}} \text{ tanh}^3\xi, \end{aligned}$$

with $\xi = \pm\frac{1}{2\sqrt{47}}\ x + c_2 t + \delta$.

For $\sigma = 16/\sqrt{73}$:

$$\begin{aligned} u(x,t) &= \frac{2\,(30 \mp 5329 c_2)}{73\sqrt{73}} \pm \frac{75}{73\sqrt{73}} \text{ tanh}\xi \\ &- \frac{60}{73\sqrt{73}} \text{ tanh}^2\xi \pm \frac{15}{73\sqrt{73}} \text{ tanh}^3\xi, \end{aligned}$$

with $\xi = \pm\frac{1}{2\sqrt{73}}\ x + c_2 t + \delta$.

- Three-dimensional modified Korteweg-de Vries equation:

$$u_t + 6u^2 u_x + u_{xyz} = 0.$$

Solitary wave solution:

$$u(x, y, z, t) = \pm\sqrt{c_2 c_3}\,\mathrm{sech}\left[\mathrm{c_1 x + c_2 y + c_3 z - c_1 c_2 c_3 t} + \delta\right].$$

- The Boussinesq (wave) equation:

$$u_{tt} - \beta u_{2x} + 3uu_{2x} + 3{u_x}^2 + \alpha u_{4x} = 0,$$

or written as a first-order system (v auxiliary variable):

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + \beta u_x - 3uu_x - \alpha u_{3x} &= 0. \end{aligned}$$

Solitary wave solution:

$$\begin{aligned} u(x, t) &= \frac{\beta c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2\left[\mathrm{c_1 x + c_2 t} + \delta\right] \\ v(x, t) &= b_0 + 4\alpha c_1 c_2 \tanh^2\left[\mathrm{c_1 x + c_2 t} + \delta\right]. \end{aligned}$$

- The Broer-Kaup system:

$$\begin{aligned} u_{ty} + 2(uu_x)_y + 2v_{xx} - u_{xxy} &= 0, \\ v_t + 2(uv)_x + v_{xx} &= 0. \end{aligned}$$

Solitary wave solution:

$$\begin{aligned} u(x, t) &= -\frac{c_3}{2c_1} + c_1 \tanh\left[\mathrm{c_1 x + c_2 y + c_3 t} + \delta\right] \\ v(x, t) &= c_1 c_2 - c_1 c_2 \tanh^2\left[\mathrm{c_1 x + c_2 y + c_3 t} + \delta\right] \end{aligned}$$

Typical Examples of DDEs (lattices)

- The Toda lattice:

$$\ddot{u}_n = (1 + \dot{u}_n)\,(u_{n-1} - 2u_n + u_{n+1})\,.$$

Solitary wave solution:

$$u_n(t) = a_0 \pm \sinh(c_1)\, \tanh\left[\mathrm{c_1 n} \pm \mathrm{\sinh(c_1)\, t} + \delta\right].$$

- The Volterra lattice:

$$\begin{aligned} \dot{u}_n &= u_n(v_n - v_{n-1}) \\ \dot{v}_n &= v_n(u_{n+1} - u_n). \end{aligned}$$

Solitary wave solution:

$$\begin{aligned} u_n(t) &= -c_2\, \coth(c_1) + c_2 \tanh\left[\mathrm{c_1 n + c_2 t} + \delta\right] \\ v_n(t) &= -c_2\, \coth(c_1) - c_2 \tanh\left[\mathrm{c_1 n + c_2 t} + \delta\right]. \end{aligned}$$

- The Relativistic Toda lattice:

$$\begin{aligned} \dot{u}_n &= (1 + \alpha u_n)(v_n - v_{n-1}) \\ \dot{v}_n &= v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}). \end{aligned}$$

Solitary wave solution:

$$\begin{aligned} u_n(t) &= -c_2\, \coth(c_1) - \frac{1}{\alpha} + c_2 \tanh\left[\mathrm{c_1 n + c_2 t} + \delta\right] \\ v_n(t) &= \frac{c_2\, \coth(c_1)}{\alpha} - \frac{c_2}{\alpha} \tanh\left[\mathrm{c_1 n + c_2 t} + \delta\right]. \end{aligned}$$

Algorithm for Tanh Solutions for PDE system

Given is a system of PDEs of order n

$$\Delta(\mathbf{u}(\mathbf{x}), \mathbf{u}'(\mathbf{x}), \mathbf{u}''(\mathbf{x}), \cdots \mathbf{u}^{(n)}(\mathbf{x})) = \mathbf{0}.$$

Dependent variable **u** has components u_i (or $u, v, w, ...$)
Independent variable **x** has components x_i (or x, y, z, t)

Step 1:

- Seek solution of the form $u_i(\mathbf{x}) = U_i(T)$, with

  $$T = \tanh\left[\mathrm{c_1x + c_2y + c_3z + c_4t}\right] = \tanh\xi.$$

- Observe $\cosh^2\xi - \sinh^2\xi = 1$, $(\tanh\xi)' = 1 - \tanh^2\xi$ or $T' = 1 - T^2$.

- Repeatedly apply the operator rule

  $$\frac{\partial \bullet}{\partial x_i} \rightarrow c_i(1 - T^2)\frac{d\bullet}{dT}$$

  This produces a coupled system of Legendre equations of type

  $$\mathbf{P}(T, U_i, U_i', \ldots, U_i^{(n)}) = 0$$

  for $U_i(T)$.

- Example: For Boussinesq system

  $$\begin{aligned} u_t + v_x &= 0 \\ v_t + \beta u_x - 3uu_x - \alpha u_{3x} &= 0, \end{aligned}$$

  we obtain after cancelling common factors $1 - T^2$

  $$c_2U' + c_1V' = 0$$

  $$\begin{aligned} &c_2V' + \beta c_1U' - 3c_1UU' \\ &+\alpha c_1^3\left[2(1 - 3T^2)U' + 6T(1 - T^2)U'' - (1 - T^2)^2U'''\right] = 0 \end{aligned}$$

Step 2:

- Seek polynomial solutions

$$U_i(T) = \sum_{j=0}^{M_i} a_{ij} T^j$$

Balance the highest power terms in T to determine M_i.

- Example: Powers for Boussinesq system

$$M_1 - 1 = M_2 - 1, \quad 2M_1 - 1 = M_1 + 1$$

gives $M_1 = M_2 = 2$.

Hence, $U_1(T) = a_{10} + a_{11}T + a_{12}T^2, \quad U_2(T) = a_{20} + a_{21}T + a_{22}T^2$.

Step 3:

- Determine the algebraic system for the unknown coefficients a_{ij} by balancing the coefficients of the various powers of T.

- Example: Boussinesq system

$$\begin{aligned}
&a_{11}\, c_1\, (3a_{12} + 2\alpha\, c_1^2) = 0\\
&a_{12}\, c_1\, (a_{12} + 4\alpha\, c_1^2) = 0\\
&a_{21}\, c_1 + a_{11}\, c_2 = 0\\
&a_{22}\, c_1 + a_{12}\, c_2 = 0\\
&\beta a_{11}\, c_1 - 3a_{10}\, a_{11}\, c_1 + 2\alpha a_{11}\, c_1^3 + a_{21}\, c_2 = 0\\
&-3a_{11}^2\, c_1 + 2\beta\, a_{12}\, c_1 - 6a_{10}\, a_{12}\, c_1 + 16\alpha\, a_{12}\, c_1^3 + 2a_{22}\, c_2 = 0.
\end{aligned}$$

Step 4:

- Solve the nonlinear algebraic system with parameters.

  Reject complex solutions? Test the solutions.

- Example: Solution for Boussinesq case

$$\begin{aligned} a_{10} &= \frac{\beta c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} \\ a_{11} &= 0 \\ a_{12} &= -4\alpha c_1^2 \\ a_{20} &= \text{free} \\ a_{21} &= 0 \\ a_{22} &= 4\alpha c_1 c_2. \end{aligned}$$

Step 5:

- Return to the original variables.

  Test the final solution in the original equations

- Example: Solitary wave solution for Boussinesq system:

$$\begin{aligned} u(x,t) &= \frac{\beta c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2\left[\mathrm{c_1 x + c_2 t} + \delta\right] \\ v(x,t) &= a_{20} + 4\alpha c_1 c_2 \tanh^2\left[\mathrm{c_1 x + c_2 t} + \delta\right]. \end{aligned}$$

Algorithm for Sech Solutions for PDE system

Given is a system of PDEs of order n

$$\Delta(\mathbf{u}(\mathbf{x}), \mathbf{u}'(\mathbf{x}), \mathbf{u}''(\mathbf{x}), \cdots \mathbf{u}^{(n)}(\mathbf{x})) = \mathbf{0}.$$

Dependent variable **u** has components u_i (or $u, v, w, ...$)

Independent variable **x** has components x_i (or x, y, z, t)

Step 1:

- Seek solution of the form $u_i(\mathbf{x}) = U_i(S)$, with

$$S = \operatorname{sech}[\mathrm{c_1x + c_2y + c_3z + c_4t}] = \tanh \xi.$$

- Observe $(\operatorname{sech} \xi)' = -\tanh \xi \operatorname{sech} \xi$ or $S' = -TS = -\sqrt{1-S^2}\, S$. Also, $\cosh^2 \xi - \sinh^2 \xi = 1$, hence, $T^2 + S^2 = 1$ and $\frac{dT}{dS} = -\frac{S}{T}$.

- Repeatedly apply the operator rule

$$\frac{\partial \bullet}{\partial x_i} \to -c_i \sqrt{1-S^2} S \frac{d\bullet}{dS}$$

  This produces a coupled system of Legendre type equations of type

$$\mathbf{P}(S, U_i, U_i', \ldots, U_i^{(m)}) + \sqrt{1-S^2}\, \mathbf{Q}(S, U_i, U_i', \ldots, U_i^{(n)}) = 0$$

  for $U_i(S)$.

  For every equation one **must** have $P_i = 0$ or $Q_i = 0$. Only odd derivatives produce the extra factor $\sqrt{1-S^2}$.

  Conclusion: The total number of derivatives in each term in the given system should be either even or odd. No mismatch is allowed.

- Example: For the 3D mKdV equation

$$u_t + 6u^2 u_x + u_{xyz} = 0.$$

we obtain after cancelling a common factor $-\sqrt{1-S^2}\,S$

$$c_4U'+6c_1U^2U'+c_1c_2c_3[(1-6S^2)U'+3S(1-2S^2)U''+S^2(1-S^2)U'''] = 0$$

Step 2:

- Seek polynomial solutions

  $$U_i(S) = \sum_{j=0}^{M_i} a_{ij}S^j$$

  Balance the highest power terms in S to determine M_i.

- Example: Powers for the 3D mKdV case

  $$3M_1 - 1 = M_1 + 1$$

  gives $M_1 = 1$. Hence, $U(S) = a_{10} + a_{11}S$.

Step 3:

- Determine the algebraic system for the unknown coefficients a_{ij} by balancing the coefficients of the various powers of S.

- Example: System for 3D mKdV case

  $$\begin{aligned} &a_{11}c_1\,(a_{11}^2 - c_2\,c_3) = 0\\ &a_{11}\,(6a_{10}^2\,c_1 + c_1\,c_2\,c_3 + c_4) = 0\\ &a_{10}\,a_{11}^2\,c_1 == 0 \end{aligned}$$

Step 4:

- Solve the nonlinear algebraic system with parameters.

  Reject complex solutions? Test the solutions.

- Example: Solution for 3D mKdV case

$$
\begin{aligned}
a_{10} &= 0 \\
a_{11} &= \pm\sqrt{c_1\,c_3} \\
c_4 &= -c_1\,c_2\,c_3
\end{aligned}
$$

Step 5:

- Return to the original variables.

  Test the final solution in the original equations

- Example: Solitary wave solution for the 3D mKdV equation

$$u(x,y,z,t) = \pm\sqrt{c_2\,c_3}\,\mathrm{sech}(\mathrm{c_1\,x + c_2\,y + c_3\,z - c_1c_2c_3\,t}).$$

Extension: Tanh Solutions for DDE system

Given is a system of differential-difference equations (DDEs) of order n

$$\Delta(\ldots, \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, \ldots, \dot{\mathbf{u}}_n, \ldots, \mathbf{u}_n^{(m)}, \ldots) = 0.$$

Dependent variable $\mathbf{u}_n$ has components $u_{i,n}$ (or $u_n, v_n, w_n, \ldots$)

Independent variable $\mathbf{x}$ has components x_i (or n, t).

No derivatives on shifted variables are allowed!

Step 1:

- Seek solution of the form $u_{i,n}(\mathbf{x}) = U_{i,n}(T(n))$, with

  $$T(n) = \tanh\left[\mathrm{c_1 n + c_2 t} + \delta\right] = \tanh \xi.$$

- Note that the argument T depends on n. Complicates matters.

- Repeatedly apply the operator rule on $u_{i,n}$

  $$\frac{\partial \bullet}{\partial t} \rightarrow c_2(1-T^2)\frac{d\bullet}{dT}$$

  This produces a coupled system of Legendre equations of type

  $$\mathbf{P}(T, U_{i,n}, U'_{i,n}, \ldots) = 0$$

  for $U_{i,n}(T)$.

- Example: Toda lattice

  $$\ddot{u}_n = (1+\dot{u}_n)\left(u_{n-1} - 2u_n + u_{n+1}\right).$$

  transforms into

  $$\begin{aligned} & c_2^2(1-T^2)\left[2TU'_n - (1-T^2)U''_n\right] \\ & + \left[1 + c_2(1-T^2)U'_n\right]\left[U_{n-1} - 2U_n + U_{n+1}\right] = 0 \end{aligned}$$

Step 2:

- Seek polynomial solutions

$$U_{i,n}(T(n)) = \sum_{j=0}^{M_i} a_{ij} T(n)^j$$

For $U_{n+p},\ p \neq 0$, there is a phase shift:

$$U_{i,n\pm p}\left(T(n \pm p) = \sum_{j=0}^{M_i} a_{i,j}\left[T(n+p)\right]^j = \sum_{j=0}^{M_i} a_{i,j}\left[\frac{T(n) \pm \ \tanh(\mathrm{pc}_1)}{1 \pm T(n)\tanh(\mathrm{pc}_1)}\right]^j\right.$$

Balance the highest power terms in $T(n)$ to determine M_i.

- Example: Powers for Toda lattice

$$2M_1 - 1 = M_1 + 1$$

gives $M_1 = 1$.

Hence,

$$\begin{aligned} U_n(T(n)) &= a_{10} + a_{11}T(n) \\ U_{n-1}(T(n-1)) &= a_{10} + a_{11}T(n-1) = a_{10} + a_{11}\frac{T(n) - \ \tanh(\mathrm{c}_1)}{1 - T(n)\tanh(\mathrm{c}_1)} \\ U_{n+1}(T(n+1)) &= a_{10} + a_{11}T(n+1) = a_{10} + a_{11}\frac{T(n) + \ \tanh(\mathrm{c}_1)}{1 + T(n)\tanh(\mathrm{c}_1)}. \end{aligned}$$

Step 3:

- Determine the algebraic system for the unknown coefficients a_{ij} by balancing the coefficients of the various powers of $T(n)$.

- Example: Algebraic system for Toda lattice

$$c_2^2 - \ \tanh^2(\mathrm{c}_1) - \mathrm{a}_{11}\mathrm{c}_2\ \tanh^2(\mathrm{c}_1) = 0, \quad \mathrm{c}_2 - \mathrm{a}_{11} = 0$$

Step 4:

- Solve the nonlinear algebraic system with parameters.

  Reject complex solutions? Test the solutions.
- Example: Solution of algebraic system for Toda lattice

$$a_{10} = \text{free}, \quad a_{11} = c_2 = \pm \sinh(c_1)$$

Step 5:

- Return to the original variables.

  Test the final solution in the original equations
- Example: Solitary wave solution for Toda lattice:

$$u_n(t) = a_0 \pm \sinh(c_1) \tanh\left[\mathrm{c_1 n \pm \sinh(c_1)\, t} + \delta\right].$$

Example: System of DDEs: Relativistic Toda lattice

$$\begin{aligned}\dot{u}_n &= (1+\alpha u_n)(v_n - v_{n-1})\\ \dot{v}_n &= v_n(u_{n+1}-u_n+\alpha v_{n+1}-\alpha v_{n-1}).\end{aligned}$$

Step 1: Change of variables

$$u_n(x,t) = U_n(T(n)), \quad v_n(x,t) = V_n(T(n)),$$

with

$$T(n) = \tanh\left[\mathrm{c_1 n + c_2 t} + \delta\right] = \tanh\xi.$$

gives

$$\begin{aligned}&c_2(1-T^2)U_n' - (1+\alpha U_n)(V_n - V_{n-1}) = 0\\ &c_2(1-T^2)V_n' - V_n(U_{n+1}-U_n+\alpha V_{n+1}-\alpha V_{n-1}) = 0.\end{aligned}$$

Step 2: Seek polynomial solutions

$$\begin{aligned}U_n(T(n)) &= \sum_{j=0}^{M_1} a_{1j}T(n)^j\\ V_n(T(n)) &= \sum_{j=0}^{M_2} a_{2j}T(n)^j.\end{aligned}$$

Balance the highest power terms in $T(n)$ to determine M_1, and M_2 :

$$M_1+1 = M_1+M_2, \qquad M_2+1 = M_1+M_2$$

gives $M_1 = M_2 = 1$.

Hence,

$$U_n = a_{10} + a_{11}T(n), \quad V_n = a_{20} + a_{21}T(n).$$

Step 3: Algebraic system for a_{ij} :

$$\begin{aligned}
-a_{11}\, c_2 + a_{21}\ \tanh(\mathrm{c}_1) + \alpha\, \mathrm{a}_{10}\, \mathrm{a}_{21}\ \tanh(\mathrm{c}_1) &= 0 \\
a_{11}\ \tanh(\mathrm{c}_1)\,(\alpha\, \mathrm{a}_{21} + \mathrm{c}_2) &= 0 \\
-a_{21}\, c_2 + a_{11}\, a_{20}\ \tanh(\mathrm{c}_1) + 2\alpha\, \mathrm{a}_{20}\, \mathrm{a}_{21}\ \tanh(\mathrm{c}_1) &= 0 \\
\tanh(\mathrm{c}_1)\,(\mathrm{a}_{11}\, \mathrm{a}_{21} + 2\alpha\, \mathrm{a}_{21}^2 - \mathrm{a}_{11}\, \mathrm{a}_{20}\ \tanh(\mathrm{c}_1)) &= 0 \\
a_{21}\ \tanh^2(\mathrm{c}_1)\,(\mathrm{c}_2 - \mathrm{a}_{11}) &= 0
\end{aligned}$$

Step 4: Solution of the algebraic system

$$a_{10} = -c_2 \coth(c_1) - \frac{1}{\alpha}, \quad a_{11} = c_2, \quad a_{20} = \frac{c_2\ \coth(c_1)}{\alpha}, \quad a_{21} = -\frac{c_2}{\alpha}.$$

Step 5: Solitary wave solution in original variables:

$$\begin{aligned}
u_n(t) &= -c_2\ \coth(c_1) - \frac{1}{\alpha} + c_2 \tanh\left[\mathrm{c}_1\mathrm{n} + \mathrm{c}_2\mathrm{t} + \delta\right] \\
v_n(t) &= \frac{c_2\ \coth(c_1)}{\alpha} - \frac{c_2}{\alpha} \tanh\left[\mathrm{c}_1\mathrm{n} + \mathrm{c}_2\mathrm{t} + \delta\right].
\end{aligned}$$

Solving/Analyzing Systems of Algebraic Equations with Parameters

Class of fifth-order evolution equations with parameters:

$$u_t + \alpha\gamma^2 u^2 u_x + \beta\gamma u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0.$$

Well-Known Special cases

Lax case: $\alpha = \frac{3}{10}, \beta = 2, \gamma = 10$. **Two** solutions:

$$u(x,t) = 4c_1^2 - 6c_1^2 \tanh^2\left[\mathrm{c_1 x - 56c_1^5 t} + \delta\right]$$

and

$$u(x,t) = a_0 - 2c_1^2 \tanh^2\left[\mathrm{c_1 x - 2(15a_0^2 c_1 - 40a_0 c_1^3 + 28c_1^5)t} + \delta\right]$$

where a_0 is arbitrary.

Sawada-Kotera case: $\alpha = \frac{1}{5}, \beta = 1, \gamma = 5$. **Two** solutions:

$$u(x,t) = 8c_1^2 - 12c_1^2 \tanh^2\left[\mathrm{c_1 x - 16c_1^5 t} + \delta\right]$$

and

$$u(x,t) = a_0 - 6c_1^2 \tanh^2\left[\mathrm{c_1 x - (5a_0^2 c_1 - 40a_0 c_1^3 + 76c_1^5)t} + \delta\right]$$

where a_0 is arbitrary.

Kaup-Kupershmidt case: $\alpha = \frac{1}{5}, \beta = \frac{5}{2}, \gamma = 10$. **Two** solutions:

$$u(x,t) = c_1^2 - \frac{3}{2}c_1^2 \tanh^2\left[\mathrm{c_1 x - c_1^5 t} + \delta\right]$$

and

$$u(x,t) = 8c_1^2 - 12c^2 \tanh^2\left[\mathrm{c_1 x - 176c_1^5 t} + \delta\right],$$

no free constants!

Ito case: $\alpha = \frac{2}{9}, \beta = 2, \gamma = 3$. **One** solution:

$$u(x,t) = 20c_1^2 - 30c_1^2 \tanh^2\left[\mathrm{c_1 x - 96c_1^5 t} + \delta\right].$$

What about the General case?

Q1: Can we retrieve the special solutions?

Q2: What are the condition(s) on the parameters α, β, γ for solutions of tanh-type to **exist**?

Tanh solutions:

$$u(x,t) = a_0 + a_1 \tanh\left[\mathrm{c_1 x + c_2 t} + \delta\right] + \mathrm{a_2} \tanh^2\left[\mathrm{c_1 x + c_2 t} + \delta\right].$$

Nonlinear algebraic system must be analyzed, solved (or reduced!):

$$a_1(\alpha\gamma^2 a_2^2 + 6\gamma a_2 c_1^2 + 2\beta\gamma a_2 c_1^2 + 24 c_1^4) = 0$$

$$a_1(\alpha\gamma^2 a_1^2 + 6\alpha\gamma^2 a_0 a_2 + 6\gamma a_0 c_1^2 - 18\gamma a_2 c_1^2 - 12\beta\gamma a_2 c_1^2 - 120 c_1^4) = 0$$

$$\alpha\gamma^2 a_2^2 + 12\gamma a_2 c_1^2 + 6\beta\gamma a_2 c_1^2 + 360 c_1^4 = 0$$

$$\begin{aligned}&2\alpha\gamma^2 a_1^2 a_2 + 2\alpha\gamma^2 a_0 a_2^2 + 3\gamma a_1^2 c_1^2 + \beta\gamma a_1^2 c_1^2 + 12\gamma a_0 a_2 c_1^2\\ &-8\gamma a_2^2 c_1^2 - 8\beta\gamma a_2^2 c_1^2 - 480 a_2 c_1^4 = 0\end{aligned}$$

$$a_1(\alpha\gamma^2 a_0^2 c_1 - 2\gamma a_0 c_1^3 + 2\beta\gamma a_2 c_1^3 + 16 c_1^5 + c_2) = 0$$

$$\begin{aligned}&\alpha\gamma^2 a_0 a_1^2 c_1 + \alpha\gamma^2 a_0^2 a_2 c_1 - \gamma a_1^2 c_1^3 - \beta\gamma a_1^2 c_1^3 - 8\gamma a_0 a_2 c_1^3 + 2\beta\gamma a_2^2 c_1^3\\ &+136 a_2 c_1^5 + a_2 c_2 = 0\end{aligned}$$

Unknowns: a_0, a_1, a_2.

Parameters: $c_1, c_2, \alpha, \beta, \gamma$.

Solve and **Reduce** cannot be used on the whole system!

Strategy to Solve/Reduce Nonlinear Systems

Assumptions:

- All $c_i \neq 0$
- Parameters $(\alpha, \beta, \gamma, ...)$ are nonzero. Otherwise the highest powers M_i may change.
- All $a_{j\,M_i} \neq 0$. Coefficients of highest power in U_i are present.
- Solve for a_{ij}, then c_i, then find conditions on parameters.

Strategy followed by hand:

- Solve all linear equations in a_{ij} first (cost: branching). Start with the ones without parameters. Capture constraints in the process.
- Solve linear equations in c_i if they are free of a_{ij}.
- Solve linear equations in parameters if they free of a_{ij}, c_i.
- Solve quasi-linear equations for a_{ij}, c_i, parameters.
- Solve quadratic equations for a_{ij}, c_i, parameters.
- Eliminate cubic terms for a_{ij}, c_i, parameters, without solving.
- Show remaining equations, if any.

Alternatives:

- Use (adapted) Gröbner Basis Techniques.
- Use combinatorics on coefficients $a_{ij} = 0$ or $a_{ij} \neq 0$.

Actual Solution: Two major cases:

CASE 1: $a_1 = 0$, two subcases

Subcase 1-a:

$$\begin{aligned} a_2 &= -\frac{3}{2} a_0 \\ c_2 &= c_1^3 (24 c_1^2 - \beta\gamma a_0) \end{aligned}$$

where a_0 is one of the two roots of the quadratic equation:

$$\alpha\gamma^2 a_0^2 - 8\gamma a_0 c_1^2 - 4\beta\gamma a_0 c_1^2 + 160 c_1^4 = 0.$$

Subcase 1-b: If $\beta = 10\alpha - 1$, then

$$\begin{aligned} a_2 &= -\frac{6}{\alpha\gamma} c_1^2 \\ c_2 &= -\frac{1}{\alpha} (\alpha^2\gamma^2 a_0^2 c_1 - 8\alpha\gamma a_0 c_1^3 + 12 c_1^5 + 16\alpha c_1^5) \end{aligned}$$

where a_0 is arbitrary.

CASE 2: $a_1 \neq 0$, then

$$\alpha = \frac{1}{392}(39 + 38\beta + 8\beta^2)$$

and

$$a_2 = -\frac{168}{\gamma(3 + 2\beta)} c_1^2$$

provided β is one of the roots of

$$(104\beta^2 + 886\beta + 1487)(520\beta^3 + 2158\beta^2 - 1103\beta - 8871) = 0$$

Subcase 2-a: If $\beta^2 = -\frac{1}{104}(886\beta + 1487)$, then

$$\begin{aligned} \alpha &= -\frac{2\beta + 5}{26} \\ a_0 &= -\frac{49c_1^2(9983 + 4378\beta)}{26\gamma(8 + 3\beta)(3 + 2\beta)^2} \\ a_1 &= \pm\frac{336c_1^2}{\gamma(3 + 2\beta)} \\ a_2 &= -\frac{168c_1^2}{\gamma(3 + 2\beta)} \\ c_2 &= -\frac{364\, c_1^5\, (3851 + 1634\beta)}{6715 + 2946\beta}. \end{aligned}$$

Subcase 2-b: If $\beta^3 = \frac{1}{520}(8871 + 1103\beta - 2158\beta^2)$, then

$$\begin{aligned} \alpha &= \frac{39 + 38\beta + 8\beta^2}{392} \\ a_0 &= \frac{28\, c_1^2\, (6483 + 5529\beta + 1066\beta^2)}{(3 + 2\beta)(23 + 6\beta)(81 + 26\beta)\gamma} \\ a_1^2 &= \frac{28224\, c_1^4\, (4\beta - 1)(26\beta - 17)}{(3 + 2\beta)^2(23 + 6\beta)(81 + 26\beta)\gamma^2} \\ a_2 &= -\frac{168c_1^2}{\gamma(3 + 2\beta)} \\ c_2 &= -\frac{8\, c_1^5\, (1792261977 + 1161063881\beta + 188900114\beta^2)}{959833473 + 632954969\beta + 105176786\beta^2}. \end{aligned}$$

Implementation Issues – Software Demo – Future Work

- Demonstration of Mathematica package for tanh/sech methods.
- Long term goal: Develop PDESolve for closed form solutions of nonlinear PDEs and DDEs.
- Implement various methods: Lie symmetry (similarity) methods.
- Look at other types of explicit solutions involving
  - hyperbolic functions $\sinh, \cosh, \tanh, \ldots$
  - other special functions.
  - complex exponentials combined with sech or tanh.
- Example: Set of ODEs from quantum field theory

$$\begin{aligned} u_{xx} &= -u + u^3 + auv^2 \\ v_{xx} &= bv + cv^3 + av(u^2 - 1). \end{aligned}$$

Try solutions ($c_2 = 0$ for ODEs)

$$u_i(x,t) = \sum_{j=0}^{M_i} a_{ij} \, \mathrm{tanh^j[c_1x + c_2t} + \delta] + \sum_{\mathrm{j=0}}^{\mathrm{N_i}} \mathrm{b_{ij}} \, \mathrm{sech^{2j+1}[c_1x + c_2t} + \delta].$$

or

$$u_i(x,t) = \sum_{j=0}^{M_i} (\tilde{a}_{ij} + \tilde{b}_{ij} \ \mathrm{sech[c_1x + c_2t} + \delta]) \ \mathrm{tanh^j[c_1x + c_2t} + \delta].$$

Solitary wave solutions:

$$\begin{aligned} u &= \pm \tanh[\sqrt{\frac{\mathrm{a^2 - c}}{\mathrm{2(a - c)}}}\mathrm{x} + \delta] \\ v &= \pm\sqrt{\frac{1 - a}{a - c}} \ \mathrm{sech}[\sqrt{\frac{\mathrm{a^2 - c}}{\mathrm{2(a - c)}}}\mathrm{x} + \delta], \end{aligned}$$

provided $b = \sqrt{\frac{a^2 - c}{2(a-c)}}$.

- Example: Nonlinear Schrödinger equation (focusing/defocusing):

$$i\,u_t + u_{xx} \pm |u|^2 u = 0.$$

Bright soliton solution (+ sign):

$$u(x,t) = \frac{k}{\sqrt{2}} \exp[i\,(\frac{c}{2}x + (k^2 - \frac{c^2}{4})t)]\ \mathrm{sech}[\mathrm{k(x - ct - x_0)}]$$

Dark soliton solution (− sign):

$$\begin{aligned} u(x,t) = \quad & \frac{1}{\sqrt{2}}\,\exp[i(Kx - (2k^2 + 3K^2 - 2Kc + \frac{c^2}{2})t)] \\ & \left\{k\ \tanh[\mathrm{k(x - ct - x_0)}] - \mathrm{i(K - \frac{c}{2})}\right\}. \end{aligned}$$

- Example: Nonlinear sine-Gordon equation (light cone coordinates):

$$u_{xt} = \sin u.$$

Setting $\Phi = u_x, \quad \Psi = \cos(u) - 1$, gives

$$\begin{aligned} \Phi_{xt} - \Phi - \Phi\,\Psi &= 0 \\ 2\Psi + \Psi^2 + \Phi_t^2 &= 0. \end{aligned}$$

Solitary wave solution (kink):

$$\begin{aligned} \Phi &= u_x = \pm\frac{1}{\sqrt{-c}}\ \mathrm{sech}[\frac{1}{\sqrt{-\mathrm{c}}}(\mathrm{x - ct}) + \delta], \\ \Psi &= \cos(u) - 1 = 1 - 2\ \mathrm{sech}^2[\frac{1}{\sqrt{-\mathrm{c}}}(\mathrm{x - ct}) + \delta], \end{aligned}$$

in final form:

$$u(x,t) = \pm 4\ \arctan\left(\exp(\frac{1}{\sqrt{-c}}(x - ct) + \delta)\right).$$

- Example: Coupled nonlinear Schrödinger equations:

  $$\begin{aligned} i\, u_t &= u_{xx} + u(|u|^2 + h|v|^2) \\ i\, v_t &= v_{xx} + v(|v|^2 + h|u|^2) \end{aligned}$$

  Seek particular solutions

  $$\begin{aligned} u(x,t) &= a \tanh(\mu \mathrm{x}) \exp(\mathrm{iAt}) \\ v(x,t) &= b\, \mathrm{sech}(\mu \mathrm{x}) \exp(\mathrm{iBt}). \end{aligned}$$

- Seek solutions $u(x,t) = U(F(\xi))$, where derivatives of $F(\xi)$ are polynomial in F.

  Now,

  $$F'(\xi) = 1 - F^2(\xi) \quad \longrightarrow \quad F = \tanh(\xi).$$

  Other choices are possible.

- Add the constraining differential equations to the system of PDEs directly.

- Why are tanh and sech solutions so prevalent?

- Other applications:

  Computation of conservation laws, symmetries, first integrals, etc. leading to **linear** parameterized systems for unknowns coefficients (see InvariantsSymmetries by Göktaş and Hereman).