SYMBOLIC COMPUTATIONS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS FROM SOLITON THEORY

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I. INTRODUCTION

- MACSYMA - symbolic manipulation programs

- Hirota's Bilinear Method
  exact **soliton** solutions of nonlinear
  - evolution equations
  - wave equations
  - coupled systems

- Hirota's conditions

- Algorithm

- Examples:

  - Korteweg-de Vries equation (KdV)
  - Kadomtsev-Petviashvili equation (KP)
  - Modified Korteweg-de Vries equation (MKdV)
  - Higher order bilinear systems
  - Bilinear equation with a parameter

- Classes of bilinear equations

- Conclusion

II. HIROTA'S BILINEAR METHOD

Hirota's method requires:

- a clever change of dependent variable
- the introduction of a bilinear differential operator
- a perturbation expansion

Example:

The Korteweg-de Vries equation

$$u_t + 6uu_x + u_{3x} = 0$$

Substitute

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}$$

Integrate with respect to x

$$f f_{xt} - f_x f_t + f f_{4x} - 4 f_x f_{3x} + 3 f_{2x}^2 = 0$$

Write in *bilinear form*

$$B(f\cdot f) \overset{\text{def}}{=} \left(D_x D_t + D_x^4\right)(f\cdot f) = 0$$

Introduce the bilinear operator

$$D_x^m D_t^n (f \cdot g) = (\partial x - \partial x')^m (\partial t - \partial t')^n f(x,t) g(x',t')|_{x'=x, t'=t}$$

Introduce the expansion (book keeping parameter ϵ)

$$f = 1 + \sum_{n=1}^{\infty} \epsilon^n f_n$$

Perturbation scheme (equate powers in ϵ to zero)

$$\begin{aligned}
O(\epsilon^0) &: B(1 \cdot 1) = 0 \\
O(\epsilon^1) &: B(1 \cdot f_1 + f_1 \cdot 1) = 0 \\
O(\epsilon^2) &: B(1 \cdot f_2 + f_1 \cdot f_1 + f_2 \cdot 1) = 0 \\
O(\epsilon^3) &: B(1 \cdot f_3 + f_1 \cdot f_2 + f_2 \cdot f_1 + f_3 \cdot 1) = 0 \\
O(\epsilon^4) &: B(1 \cdot f_4 + f_1 \cdot f_3 + f_2 \cdot f_2 + f_3 \cdot f_1 + f_4 \cdot 1) = 0 \\
O(\epsilon^n) &: B(\sum_{j=0}^{n} f_j \cdot f_{n-j}) = 0 \,, \qquad \text{with } \; f_0 = 1
\end{aligned}$$

If the original PDE admits a N-soliton solution then the expansion will truncate at level $n = N$ provided

$$f_1 = \sum_{i=1}^{N} \exp \theta_i = \sum_{i=1}^{N} \exp (k_i x - \omega_i t + \delta_i)$$

k_i, ω_i and δ_i are constants

Example: three-soliton solution ($N = 3$)

Determine the dispersion law

$$\omega_i = k_i^3 \,, \qquad i = 1, 2, 3$$

Terms generated by $B(f_1 {\cdot} f_1)$ justify

$$\begin{aligned} f_2 &= a_{12} \exp(\theta_1 + \theta_2) + a_{13} \exp(\theta_1 + \theta_3) + a_{23} \exp(\theta_2 + \theta_3) \\ &= a_{12} \exp \left[(k_1 + k_2) x - (\omega_1 + \omega_2) t + (\delta_1 + \delta_2) \right] \\ &+ a_{13} \exp \left[(k_1 + k_3) x - (\omega_1 + \omega_3) t + (\delta_1 + \delta_3) \right] \\ &+ a_{23} \exp \left[(k_2 + k_3) x - (\omega_2 + \omega_3) t + (\delta_2 + \delta_3) \right] \end{aligned}$$

Calculate the constants a_{12}, a_{13} and a_{23}

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}\,, \qquad i,j = 1,2,3, \qquad i < j$$

$B(f_1 \cdot f_2 + f_2 \cdot f_1)$ motivates

$$\begin{aligned} f_3 &= b_{123} \exp(\theta_1 + \theta_2 + \theta_3) \\ &= b_{123} \exp\left[(k_1+k_2+k_3)x - (\omega_1+\omega_2+\omega_3)t + (\delta_1+\delta_2+\delta_3)\right] \end{aligned}$$

with

$$b_{123} = a_{12}a_{13}a_{23} = \frac{(k_1 - k_2)^2(k_1 - k_3)^2(k_2 - k_3)^2}{(k_1 + k_2)^2(k_1 + k_3)^2(k_2 + k_3)^2}$$

Subsequently, $f_i = 0$ for $i \geq 4$

Set $\epsilon = 1$

$$\begin{aligned} f &= 1 + \exp\theta_1 + \exp\theta_2 + \exp\theta_3 \\ &+ a_{12}\exp(\theta_1 + \theta_2) + a_{13}\exp(\theta_1 + \theta_3) + a_{23}\exp(\theta_2 + \theta_3) \\ &+ b_{123}\exp(\theta_1 + \theta_2 + \theta_3) \end{aligned}$$

Coupled Equations

Example: the modified Korteweg-de Vries equation

$$u_t + 6u^2 u_x + u_{3x} = 0$$

Introducing the potential w by $u = w_x$

$$w_{tx} + 6w_x^2 w_{xx} + w_{4x} = 0$$

Integrate with respect to x

$$w_t + 2w_x^3 + w_{3x} = 0$$

Set

$$w(x,t) = -2i \arctan\left(\frac{G}{F}\right)$$

Bilinear system

$$(F^2+G^2)[(D_x^3+D_t)(G\cdot F)] - 3D_x(G\cdot F)[D_x^2(F\cdot F+G\cdot G)] = 0$$

Require that

$$\begin{aligned}(D_x^3 + D_t)(G\cdot F) &= 0\\ D_x^2(F\cdot F + G\cdot G) &= 0\end{aligned}$$

Perform a complex rotation

$$f = F + iG,\ g = F - iG, \quad w = \ln(f/g)$$

$$\begin{aligned} P_1(D_x, D_t)(f \cdot g) &\stackrel{\text{def}}{=} (D_x^3 + D_t)(f \cdot g) = 0 \\ P_2(D_x, D_t)(f \cdot g) &\stackrel{\text{def}}{=} D_x^2(f \cdot g) = 0 \end{aligned}$$

Take

$$\begin{aligned} f &= 1 + \sum_{n=1}^{\infty} \epsilon^n f_n \\ g &= 1 + \sum_{n=1}^{\infty} \epsilon^n g_n \end{aligned}$$

Method of solution is similar

III. HIROTA'S CONDITION

Single Bilinear equation

$$P(D_x, D_t)(f \cdot f) = 0$$

P is an arbitrary polynomial

Example: KdV equation

$$P(D_x, D_t) = D_x D_t + D_x^4$$

If P satisfies

$$\begin{aligned} P(D_x, D_t) &= P(-D_x, -D_t) \\ P(0,0) &= 0 \end{aligned}$$

then the equation always has a two-soliton solution

- Single soliton solution

$$f = 1 + e^{\theta}\ , \qquad \theta = kx - \omega t + \delta$$

k, ω and δ are constants and

$$P(k, -\omega) = 0$$

Substituting f into

$$\begin{aligned} u(x,t) &= 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} \\ &= 2(\frac{f_{xx}f - f_x^2}{f^2}) \end{aligned}$$

Denote $k = 2K$, to get the well-known solitary wave solution

$$u = 2K^2 \mathrm{sech}^2 K(x - 4K^2 t + \delta)\ ,$$

- Two-soliton solution

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}$$

$$\theta_i = k_i x - \omega_i t + \delta_i$$

$$P(k_i, -\omega_i) = 0 \quad \text{or} \quad \omega_i = k_i^3 \ , \quad i = 1, 2$$

$$a_{12} = -\frac{P(k_1 - k_2, -\omega_1 + \omega_2)}{P(k_1 + k_2, -\omega_1 - \omega_2)}$$

Example: for the KdV equation

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$$

Selecting

$$e^{\delta_i} = \frac{c_i^2}{k_i} e^{k_i x - \omega_i t + \Delta_i}$$

$$\tilde{f} = \frac{1}{4} f e^{-\frac{1}{2}(\tilde{\theta}_1 + \tilde{\theta}_2)}$$

$$\tilde{\theta}_i = k_i x - \omega_i t + \Delta_i$$

$$c_i^2 = \left(\frac{k_2 + k_1}{k_2 - k_1}\right) k_i$$

One obtains upon return to u

$$\begin{aligned} u(x,t) &= \tilde{u}(x,t) = 2\frac{\partial^2 \ln \tilde{f}(x,t)}{\partial x^2} \\ &= \left(\frac{k_2^2 - k_1^2}{2}\right)\left(\frac{k_2^2 \mathrm{cosech}^2 \frac{\tilde{\theta}_2}{2} + k_1^2 \mathrm{sech}^2 \frac{\tilde{\theta}_1}{2}}{(k_2 \coth \frac{\tilde{\theta}_2}{2} - k_1 \tanh \frac{\tilde{\theta}_1}{2})^2}\right) \end{aligned}$$

- For the general N-soliton solution

$$f = \sum_{\mu=0,1} \exp\left[\sum_{i<j}^{(N)} A_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \mu_i\theta_i\right]$$

$$a_{ij} = \exp A_{ij} = -\frac{P(k_i - k_j, -\omega_i + \omega_j)}{P(k_i + k_j, -\omega_i - \omega_j)}$$

Additional condition for $P(D_x, D_t)$

$$\begin{aligned} S[P,n] \; = \; & \sum_{\sigma=\pm 1} P\left(\sum_{i=1}^{n} \sigma_i k_i, -\sum_{i=1}^{n} \sigma_i\omega_i\right) \\ & \times \prod_{i<j}^{(n)} P(\sigma_i k_i - \sigma_j k_j, -\sigma_i\omega_i + \sigma_j\omega_j)\sigma_i\sigma_j = 0 \; , \\ & \text{for} \;\; n = 2, \ldots, N \end{aligned}$$

Coupled bilinear equations

$$P_1(D_x, D_t)(f \cdot g) \;=\; 0$$
$$P_2(D_x, D_t)(f \cdot g) \;=\; 0$$

If P_1 and P_2 satisfy

$$P_i(-D_x, -D_t) \;=\; (-1)^i P_i(D_x, D_t)\;, \qquad i = 1, 2$$
$$P_2(0, 0) \;=\; 0$$

then the equations always have at least a two-soliton solution

- One-soliton solution

$$f \;=\; 1 + e^{\theta}$$

$$g \;=\; 1 - e^{\theta}$$

with

$$\theta = kx - \omega t + \delta$$

k, ω and δ are constants and

$$P_1(k, -\omega) = 0$$

- Two-soliton solution

$$\begin{aligned} f &= 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2} \\ g &= 1 - e^{\theta_1} - e^{\theta_2} + a'_{12} e^{\theta_1 + \theta_2} \end{aligned}$$

where $\theta_i = k_i x - \omega_i t + \delta_i$

and

$$P_1(k_i, -\omega_i) = 0 \ , \qquad i = 1, 2$$

$$a'_{12} = a_{12} = \frac{P_2(k_1 - k_2, -\omega_1 + \omega_2)}{P_2(k_1 + k_2, -\omega_1 - \omega_2)}$$

- For the general N-soliton solution

$$f = \sum_{\mu=0,1} \exp\left[\sum_{i<j}^{(N)} A_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \mu_i(\theta_i + \frac{\sqrt{-1}\pi}{2})\right]$$

$$g = \sum_{\mu=0,1} \exp\left[\sum_{i<j}^{(N)} A_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \mu_i(\theta_i - \frac{\sqrt{-1}\pi}{2})\right]$$

$$a_{ij} = \exp A_{ij} = \frac{P_2(k_i - k_j, -\omega_i + \omega_j)}{P_2(k_i + k_j, -\omega_i - \omega_j)}, \quad i < j$$

Required additional conditions

$$S_{\rm odd}[P_1, P_2, n] = \sum_{\sigma=\pm 1} P_1\left(\sum_{i=1}^{n} \sigma_i k_i, -\sum_{i=1}^{n} \sigma_i\omega_i\right) \sin\left(\sum_{i=1}^{n} \sigma_i \frac{\pi}{2}\right)$$
$$\times \prod_{i<j}^{(n)} P_2(\sigma_i k_i - \sigma_j k_j, -\sigma_i\omega_i + \sigma_j\omega_j) = 0$$

$$S_{\rm even}[P_2, n] = \sum_{\sigma=\pm 1} P_2\left(\sum_{i=1}^{n} \sigma_i k_i, -\sum_{i=1}^{n} \sigma_i\omega_i\right) \cos\left(\sum_{i=1}^{n} \sigma_i \frac{\pi}{2}\right)$$
$$\times \prod_{i<j}^{(n)} P_2(\sigma_i k_i - \sigma_j k_j, -\sigma_i\omega_i + \sigma_j\omega_j) = 0$$

for $n = 2, 3, \ldots, N$
and all k_i, ω_i subject to

$$P_1(k_i, -\omega_i) = 0$$

IV. ALGORITHM AND PROGRAM

- Special features
  - New expressions for Hirota's bilinear operators

$$D_x^n(f\cdot g) = \sum_{j=0}^{n} \frac{(-1)^{(n-j)} n!}{j!(n-j)!} \frac{\partial^j f}{\partial x^j} \frac{\partial^{n-j} g}{\partial x^{n-j}}$$

$$D_x^m D_t^n(f\cdot g) = \sum_{j=0}^{m} \sum_{i=0}^{n} \frac{(-1)^{(m+n-j-i)} m! n!}{j!(m-j)! i!(n-i)!} \frac{\partial^{i+j} f}{\partial t^i \partial x^j} \frac{\partial^{n+m-i-j} g}{\partial t^{n-i} \partial x^{m-j}}$$

  - Exponential function $h(x,t) = \exp(kx - \omega t + \delta)$ is defined via **gradef**

$$\frac{\partial h(x,t)}{\partial x} = k h(x,t)$$

$$\frac{\partial h(x,t)}{\partial t} = -\omega h(x,t)$$

- Program consists of 14 blocks (functions)
- MACSYMA PROGRAM

  The symbolic program calculates

– the one-, two- and three- soliton solutions
– test conditions for existence of up to a 4-soliton solution
– verifies equality of a_{ij} and b_{123} calculated with two different methods

In a batch program, user must provide

– bilinear operator B
– name: name of the PDE
– N: N-soliton solution
– test_for_3soliton: TRUE or FALSE
– check_coefficients: TRUE or FALSE
– test_for_4soliton: TRUE or FALSE

- Batch Program

```
writefile("test_kdv.out")$
loadfile("hirota_single.lsp")$
N:3$
B(f,g):=Dxt[1,1](f,g)+Dx[4](f,g)$
name:Korteweg_de_Vries$
hirota(B,name,N,true,true,true)$
closefile()$
quit()$
```

V. EXAMPLES

- The Korteweg-de Vries equation

$$u_t + 6uu_x + u_{3x} = 0$$
$$B(f,g) = Dxt[1,1](f,g) + Dx[4](f,g)$$

There is at least four-soliton solution.
The three-soliton solution is generated from

$$\begin{aligned} f &= 1 + \exp\theta_1 + \exp\theta_2 + \exp\theta_3 \\ &+ a_{12}\exp(\theta_1+\theta_2) + a_{13}\exp(\theta_1+\theta_3) + a_{23}\exp(\theta_2+\theta_3) \\ &+ b_{123}\exp(\theta_1+\theta_2+\theta_3) \end{aligned}$$

with

$$\theta_i = k_i x - \omega_i t + \delta_i\ , \qquad \omega_i = k_i^3$$
$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}\ , \qquad i,j = 1,2,3\ , \qquad i<j$$
$$b_{123} = a_{12}a_{13}a_{23} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\frac{(k_1-k_3)^2}{(k_1+k_3)^2}\frac{(k_2-k_3)^2}{(k_2+k_3)^2}$$

• The Kadomtsev-Petviashvili equation

$$(u_t + 6uu_x + u_{3x})_x + 3u_{2y} = 0$$
$$u(x,y,t) = 2\frac{\partial^2 \ln f(x,y,t)}{\partial x^2}$$

$$B(f,g) = Dxt[1,1](f,g) + Dx[4](f,g) + 3Dy[2](f,g)$$

There is at least a three-soliton solution

$$\begin{aligned} f &= 1 + \exp\theta_1 + \exp\theta_2 + \exp\theta_3 \\ &+ a_{12}\exp(\theta_1+\theta_2) + a_{13}\exp(\theta_1+\theta_3) + a_{23}\exp(\theta_2+\theta_3) \\ &+ b_{123}\exp(\theta_1+\theta_2+\theta_3)\ , \end{aligned}$$

Here, $\theta_i = k_i x + l_i y - \omega_i t$

$$\omega_i = \frac{3l_i^2 + k_i^4}{k_i}\ , \quad i = 1,2,3$$

$$a_{ij} = \frac{(k_i k_j^2 - k_i^2 k_j - l_i k_j + l_j k_i)(k_i k_j^2 - k_i^2 k_j + l_i k_j - l_j k_i)}{(k_i k_j^2 + k_i^2 k_j + l_i k_j - l_j k_i)(k_i k_j^2 + k_i^2 k_j - l_i k_j + l_j k_i)}$$

$$b_{123} = a_{12}a_{13}a_{23}$$

- For the modified Korteweg-de Vries equation

$$u_t + 6u^2 u_x + u_{3x} = 0$$

$$\begin{aligned} B1(f,g) &= Dxt[0,1](f,g) + Dx[3](f,g) \\ B2(f,g) &= Dx[2](f,g) \end{aligned}$$

There is a four-soliton solution.
For the three-soliton solution

$$\begin{aligned} f &= 1 + i\exp\theta_1 + i\exp\theta_2 + i\exp\theta_3 \\ &- a_{12}\exp(\theta_1+\theta_2) - a_{13}\exp(\theta_1+\theta_3) - a_{23}\exp(\theta_2+\theta_3) \\ &- ib_{123}\exp(\theta_1+\theta_2+\theta_3) \end{aligned}$$

$$\begin{aligned} g &= 1 - i\exp\theta_1 - i\exp\theta_2 - i\exp\theta_3 \\ &- a_{12}\exp(\theta_1+\theta_2) - a_{13}\exp(\theta_1+\theta_3) - a_{23}\exp(\theta_2+\theta_3) \\ &+ ib_{123}\exp(\theta_1+\theta_2+\theta_3) \end{aligned}$$

with $\theta_i = k_i x - \omega_i t + \delta_i$, $\qquad \omega_i = k_i^3$

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \qquad i,j = 1,2,3, \qquad i < j$$

$$b_{123} = a_{12}a_{13}a_{23} = \frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}$$

- For the bilinear system

$$\begin{aligned} B1(f,g) &= Dxt[0,1](f,g) + Dx[5](f,g) \\ B2(f,g) &= Dxt[0,2](f,g) + Dxt[5,1](f,g) \end{aligned}$$

there exists a two-soliton solution but there is no three-soliton solution.
For the two-soliton solution

$$\begin{aligned} f &= 1 + i\exp\theta_1 + i\exp\theta_2 - a_{12}\exp(\theta_1+\theta_2) \\ g &= 1 - i\exp\theta_1 - i\exp\theta_2 - a_{12}\exp(\theta_1+\theta_2) \end{aligned}$$

with $\theta_i = k_i x - \omega_i t + \delta_i$, $\qquad \omega_i = k_i^5$, $\qquad i = 1,2$

$$a_{12} = \frac{(k_1-k_2)^2(k_1^3+k_2^3)(k_1^5-k_2^5)}{(k_1+k_2)^2(k_1^3-k_2^3)(k_1^5+k_2^5)}$$

- For the bilinear system with a parameter a,

$$\begin{aligned} B1(f,g) &= Dxt[0,1](f,g) + Dx[3](f,g) \\ B2(f,g) &= Dxt[1,1](f,g) + a * Dx[4](f,g) \end{aligned}$$

there exists a two-soliton solution.
For a three-soliton solution to exist, the condition

$$72(a-1)a(2a+1)k_1k_2k_3(k_1-k_2)^2(k_1+k_2)^2 \\ (k_1-k_3)^2(k_1+k_3)^2(k_2-k_3)^2(k_2+k_3)^2 = 0$$

must be satisfied.
For the two-soliton solution

$$\begin{aligned} f &= 1 + i\exp\theta_1 + i\exp\theta_2 - a_{12}\exp(\theta_1+\theta_2) \\ g &= 1 - i\exp\theta_1 - i\exp\theta_2 - a_{12}\exp(\theta_1+\theta_2) \end{aligned}$$

with $\theta_i = k_i x - \omega_i t + \delta_i \ , \qquad \omega_i = k_i^5 \ , \qquad i = 1, 2$

$$a_{12} = \frac{(k_1-k_2)^2(ak_1^2 - k_1^2 - 2ak_1k_2 - k_1k_2 + ak_2^2 - k_2^2)}{(k_1+k_2)^2(ak_1^2 - k_1^2 + 2ak_1k_2 + k_1k_2 + ak_2^2 - k_2^2)}$$

VI. CLASSES OF BILINEAR EQUATIONS

Type I:

$$P(D_x, D_t)(f \cdot f) \overset{ex.}{=} (D_x D_t + D_x^4)(f \cdot f) = 0 \qquad \text{(KdV type)}$$

Type II:

$$\begin{aligned} P_1(D_x, D_t)(f \cdot g) &\overset{ex.}{=} (D_t + D_x^3)(f \cdot g) = 0 \\ P_2(D_x, D_t)(f \cdot g) &\overset{ex.}{=} D_x^2(f \cdot g) = 0 \qquad \text{(mKdV type)} \end{aligned}$$

Type III:

$$\begin{aligned} P_1(D_x, D_t)(g \cdot f) &\overset{ex.}{=} (D_x D_t - 1)(g \cdot f) = 0 \\ P_2(D_x, D_t)(f \cdot f - g \cdot g) &\overset{ex.}{=} D_x D_t (f \cdot f - g \cdot g) = 0 \quad \text{(SG type)} \end{aligned}$$

Type IV:

$$\begin{aligned}P_1(D_x,D_t)(g\cdot f) &\stackrel{ex.}{=} (D_x^2+iD_t)(g\cdot f)=0\\ P_2(D_x,D_t)(f\cdot f) &- P_3(D_x,D_t)(g\cdot g^*) \stackrel{ex.}{=} D_x^2(f\cdot f)-gg^*=0\\ &\quad \text{(NLS type)}\end{aligned}$$

Type V:

$$P(D_x,D_t)(f\cdot f^*) \stackrel{ex.}{=} (D_x^2+iD_t)(f\cdot f^*)=0 \qquad \text{(BO type)}$$

For Type I the form of f is as before
For Type II, the three-soliton solution follows from

$$\begin{aligned}f &= 1+i\exp\theta_1+i\exp\theta_2+i\exp\theta_3-a_{12}\exp(\theta_1+\theta_2)\\ &- a_{13}\exp(\theta_1+\theta_3)-a_{23}\exp(\theta_2+\theta_3)\\ &- ib_{123}\exp(\theta_1+\theta_2+\theta_3),\end{aligned}$$

$$\begin{aligned}g &= 1-i\exp\theta_1-i\exp\theta_2-i\exp\theta_3-a_{12}\exp(\theta_1+\theta_2)\\ &- a_{13}\exp(\theta_1+\theta_3)-a_{23}\exp(\theta_2+\theta_3)\\ &+ ib_{123}\exp(\theta_1+\theta_2+\theta_3)\end{aligned}$$

For Type III through V other forms for f and g are needed

VII. CONCLUSIONS

- use as integrability test

- use exact solutions to test numerical solution algorithms

- symbolic computation can play a significant role in classifying integrable equations

- future research work
  develop a new program that calculates soliton solutions of an entire family of coupled bilinear systems

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