Analysis of the DCS one-stage Greedy Algorothm for Common Sparse Supports

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Setup

We denote the signals by x_j, $j \in \{1, \ldots, J\}$, and assume that each signal $x_j \in \mathbb{R}^N$. Our DCS II model for joint sparsity concerns the case of multiple sparse signals that share common sparse components, but with different coefficients. For example,

$$x_j = \Psi \alpha_j,$$

where each α_j is supported only on $\Omega \subset \{1, 2, \ldots, N\}$, with $|\Omega| = K$. The matrix Ψ is orthonormal, with dimension $N \times N$ (we consider only signals sparse in an orthonormal basis).

We denote by Φ_j the measurement matrix for signal j, where Φ_j is of dimension $M \times N$, where $M < N$. We let $y_j = \Phi_j x_j = \Phi_j \Psi \alpha_j$ be the observations of signal j.

We assume that the measurement matrix Φ_j is random with i.i.d entries taken from a $\mathcal{N}(0, 1)$ distribution. Clearly, the matrix $\Phi_j \Psi$ also has i.i.d $\mathcal{N}(0, 1)$ entries, because Ψ is orthonormal. For convenience, we assume Ψ to be identity $I_{N \times N}$. The results presented can be easily extended to a more general orthonormal matrix Ψ by replacing Φ_j with $\Phi_j \Psi$.

Recovery

After gathering all of the measurements, we compute the following statistic for each $n \in \{1, \ldots, N\}$:

$$\theta_n = \frac{\sum_{j=1}^{J} \langle y_j, \phi_{j,n} \rangle^2}{J}, \tag{1}$$

where $\phi_{j,n}$ denotes column n of measurement matrix Φ_j. To estimate Ω we choose the K largest statistics θ_n. We have the following results.

Theorem 1 *Assume the $M \times N$ measurement matrices Φ_j contain i.i.d. $\mathcal{N}(0,1)$ entries and that the coefficient vectors x_j contain i.i.d. $\mathcal{N}(0,\sigma^2)$ entries. Let $y_j = \Phi_j x_j$ and let θ_n be defined as in Equation (1). The mean and variance of θ_n are given by*

$$E\theta_n = \begin{cases} m_b & if \quad n \notin \Omega \\ m_g & if \quad n \in \Omega \end{cases}$$

and

$$Var(\theta_n) = \begin{cases} \sigma_b^2 & if \quad n \notin \Omega \\ \sigma_g^2 & if \quad n \in \Omega, \end{cases}$$

where

$$\begin{aligned} m_b &= MK\sigma^2, \\ m_g &= M(M+K+1)\sigma^2, \\ \sigma_b^2 &= \frac{2MK\sigma^4}{J}(MK+3K+3M+6), \quad \textit{and} \\ \sigma_g^2 &= \frac{M\sigma^4}{J}(34MK+6K^2+28M^2+92M+48K+90+2M^3+2MK^2+4M^2K). \end{aligned}$$

Theorem 2 *The one shot algorithm recovers Ω with a probability of success p_s given by approximately*

$$p_s \approx \frac{1}{2^{N-1}}\frac{(N-K)}{\sigma_b\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[1+erf\left(\frac{x-m_b}{\sigma_b\sqrt{2}}\right)\right]^{N-K-1}\left[1-erf\left(\frac{x-m_g}{\sigma_g\sqrt{2}}\right)\right]^{K}\exp\left[-\left(\frac{x-m_b}{\sigma_b\sqrt{2}}\right)^2\right]dx.$$

Corollary 1 *The one-stage algorithm recovers Ω with probability approaching 1 as $J \to \infty$.*

Remark 1 *The mean and variance of θ_n are independent of N.*

Remark 2 *The variance of θ_n goes to zero as $J \to \infty$.*

Proof of Theorem 1

We first present a short sketch of the strategy we adopt to prove the result. The main idea is to compute the statistics of $\langle y_j, \phi_{j,n}\rangle$ up to first four moments, for $n \in \Omega$ and $n \notin \Omega$. Based on these results, we derive the mean and variance of θ_n.

We use the following ideas in our proof. Let X_1 and X_2 be two independent random variables. Define random variables Y and Z as $Y = X_1 \times X_2$ and $Z = X_1 + X_2$. Then, the p^{th} moment of Y — which we denote by $m_p(Y)$ — is given by $m_p(Y) = m_p(X_1) \times m_p(X_2)$. Furthermore, the p^{th} cumulant [1] of Z — denoted by $c_p(Y)$ — is given by $c_p(Z) = c_p(X_1) + c_p(X_2)$. When we multiply independent random variables, we work with their moments.

While working with the sum of independent random variables, we work with their cumulants. We use the standard formulae [1] to convert from moments to cumulants and vice versa.

We use the notation $X = \text{Moments}(m_1, m_2, ..., m_p)$ to keep track of the first p moments of the random variable X. Likewise, we denote $X = \text{Cumulants}(c_1, c_2, ..., c_p)$ to keep track of the first p cumulants of X. The conversion from cumulants to moments and vice versa for up to two orders is as follows:

$$\text{Cumulants}(c_1, c_2) \equiv \text{Moments}(m_1, m_2)$$

if $c_1 = m_1$, and $m_2 = c_2 + c_1^2$ (or equivalently $c_2 = m_2 - m_1^2$). The first and second cumulants correspond, respectively, to the mean and variance.

We also use the results for the moments of a Gaussian Random variable $X \sim \mathcal{N}(0, 1)$: $EX^4 = 3$ and $EX^6 = 15$.

We begin by computing statistics of operations on the columns of the matrix Φ_j. These results are presented in the form of five Lemmas.

Lemma 1 *For $1 \leq j \leq J$, $1 \leq n, l \leq N$ and $n \neq l$,*

$$E\langle \phi_{j,n}, \phi_{j,l} \rangle^2 = M.$$

Proof of Lemma: Let $\phi_{j,n}$ be the column vector $[a_1, a_2, ..., a_M]^T$, where each element in the vector is iid $\mathcal{N}(0, 1)$. Likewise, let $\phi_{j,l}$ be the column vector $[b_1, b_2, ..., b_M]^T$ where the elements are iid $\mathcal{N}(0, 1)$. We have

$$\begin{aligned} \langle \phi_{j,n}, \phi_{j,l} \rangle^2 &= (a_1 b_1 + a_2 b_2 + ... a_M b_M)^2 \\ &= \sum_{q=1}^{M} a_q^2 b_q^2 + 2 \sum_{q=1}^{M-1} \sum_{r=q+1}^{M} a_q a_r b_q b_r. \end{aligned}$$

Taking expectations,

$$\begin{aligned} E\left[\langle \phi_{j,n}, \phi_{j,l} \rangle^2\right] &= E\left[\sum_{q=1}^{M} a_q^2 b_q^2\right] + 2E\left[\sum_{q=1}^{M-1} \sum_{r=q+1}^{M} a_q a_r b_q b_r\right] \\ &= \sum_{q=1}^{M} E(a_q^2 b_q^2) + 2 \sum_{q=1}^{M-1} \sum_{r=q+1}^{M} E(a_q a_r b_q b_r) \\ &= \sum_{q=1}^{M} E(a_q^2) E(b_q^2) + 2 \sum_{q=1}^{M-1} \sum_{r=q+1}^{M} E a_q E a_r E b_q E b_r \\ &\quad \text{(because the random variables are independent)} \\ &= \sum_{q=1}^{M} (1) + 0 \\ &\quad \text{(because } E(a_q^2) = E(b_q^2) = 1 \text{ and } E(a_q) = E(b_q) = 0) \\ &= M. \end{aligned}$$

This completes the proof of the Lemma.

Lemma 2 *For $1 \le j \le J$, $1 \le n, l \le N$ and $n \ne l$,*

$$E\langle \phi_{j,n}, \phi_{j,l}\rangle^4 = 3M(M+2).$$

Proof of Lemma: As before, let $\phi_{j,n}$ be the column vector $[a_1, a_2, ..., a_M]^T$, where each element in the vector is iid $\mathcal{N}(0,1)$. Likewise, let $\phi_{j,l}$ be the column vector $[b_1, b_2, ..., b_M]^T$ where the elements are iid $\mathcal{N}(0,1)$. We have

$$\begin{aligned}
E\langle \phi_{j,n}, \phi_{j,l}\rangle^4 &= E(a_1 b_1 + a_2 b_2 + ... a_M b_M)^4 \\
&= E\left[\sum_{q=1}^{M} a_q^4 b_q^4\right] + \binom{4}{2} E\left[\sum_{q=1}^{M-1}\sum_{r=q+1}^{M} (a_q b_q)^2 (a_r b_r)^2\right] \\
&\quad + E(\text{cross terms with zero expectation}) \\
&= \sum_{q=1}^{M} E a_q^4 E b_q^4 + 6 \sum_{q=1}^{M-1}\sum_{r=q+1}^{M} E a_q^2 E b_q^2 E a_r^2 E b_r^2 \quad \text{(by independence)} \\
&= 9M + 6\frac{M(M-1)}{2} \quad (\text{because } E a_q^4 = 3 \text{ and } E a_q^2 = 1) \\
&= 3M(M+2).
\end{aligned}$$

This completes the proof of the Lemma.

Lemma 3 *For $1 \le j \le J$, $1 \le n, l, q \le N$ and unique n, l and q,*

$$E\left[\langle \phi_{j,n}, \phi_{j,l}\rangle^2 \langle \phi_{j,n}, \phi_{j,q}\rangle^2\right] = M(M+2).$$

Proof of Lemma: As before, let $\phi_{j,n}$ be the column vector $[a_1, a_2, ..., a_M]^T$, where each element in the vector is iid $\mathcal{N}(0,1)$. Likewise, let $\phi_{j,n}$ be the column vector $[b_1, b_2, ..., b_M]^T$ and $\phi_{j,q}$ be the column vector $[c_1, c_2, ..., c_M]^T$. From the statement of the Lemma,

$$\begin{aligned}
LHS &= E\left[\langle \phi_{j,n}, \phi_{j,l}\rangle^2 \langle \phi_{j,n}, \phi_{j,q}\rangle^2\right] \\
&= E\left[(a_1 b_1 + a_2 b_2 + ... a_M b_M)^2 (a_1 c_1 + a_2 c_2 + ... a_M c_M)^2\right] \\
&= E\left[\left(\sum_{r=1}^{M} a_r^2 b_r^2 + \sum_{r=1}^{M}\sum_{s=1, s\ne r}^{M} a_r b_r a_s b_s\right)\left(\sum_{r=1}^{M} a_r^2 c_r^2 + \sum_{r=1}^{M}\sum_{s=1, s\ne r}^{M} a_r c_r a_s c_s\right)\right].
\end{aligned}$$

Collecting only those terms with non-zero expectations,

$$\begin{aligned}
LHS &= E\left[\sum_{r=1}^{M} a_r^4 b_r^2 c_r^2 + \sum_{r=1}^{M}\sum_{s=1,s\neq r}^{M} a_r^2 a_s^2 b_r^2 c_s^2\right] \\
&= \sum_{r=1}^{M} Ea_r^4 Eb_r^2 Ec_r^2 + \sum_{r=1}^{M}\sum_{s=1,s\neq r}^{m} Ea_r^2 Ea_s^2 Eb_r^2 Ec_s^2 \\
&= 3M + M(M-1) \\
&= M(M+2).
\end{aligned}$$

Jhis completes the proof of the Lemma.

Lemma 4 *For $1 \leq j \leq J$, $1 \leq n, l \leq N$ and $n \neq l$,*

$$E\left[\|\phi_{j,l}\|^4 \langle \phi_{j,n}, \phi_{j,l}\rangle^2\right] = M(M+2)(M+4).$$

Proof of Lemma: As before, let $\phi_{j,n}$ be the column vector $[a_1, a_2, ..., a_M]^T$, where each element in the vector is iid $\mathcal{N}(0,1)$. Likewise, let $\phi_{j,l}$ be the column vector $[b_1, b_2, ..., b_M]^T$ where the elements are iid $\mathcal{N}(0,1)$. We have

$$\begin{aligned}
E\left[\|\phi_{j,l}\|^4 \langle \phi_{j,n}, \phi_{j,l}\rangle^2\right] &= E\left[(a_1^2 + a_2^2 + ...a_M^2)^2 (a_1 b_1 + a_2 b_2 + ...a_M b_M)^2\right] \\
&= E\left[\left(\sum_{r=1}^{M} a_r^4 + \sum_{r=1}^{M}\sum_{s=1,s\neq r}^{M} a_r^2 a_s^2\right)\left(\sum_{r=1}^{M} a_r^2 b_r^2 + \sum_{r=1}^{M}\sum_{s=1,s\neq r}^{M} a_r b_r a_s b_s\right)\right].
\end{aligned}$$

Collecting only the terms with non-zero expectations,

$$\begin{aligned}
E\left[\|\phi_{j,l}\|^4 \langle \phi_{j,n}, \phi_{j,l}\rangle^2\right] &= E\left[\sum_{r=1}^{M} a_r^6 b_r^2 + \sum_{r=1}^{M}\sum_{s=1,s\neq r}^{M} a_r^4 a_s^2 b_s^2\right. \\
&\quad \left. + 2\sum_{r=1}^{M}\sum_{s=1,s\neq r}^{M} a_r^4 a_s^2 b_r^2 + \sum_{r=1}^{M}\sum_{s=1,s\neq r}^{M}\sum_{t=1,t\neq r,s}^{M} a_r^2 a_s^2 a_t^2 b_t^2\right] \\
&= 15M + 3M(M-1) + 6M(M-1) + M(M-1)(M-2) \\
&\quad \text{(because for } X \sim \mathcal{N}(0,1),\ EX^4 = 3 \text{ and } EX^6 = 15) \\
&= M(M+2)(M+4).
\end{aligned}$$

This completes the proof of the Lemma.

Lemma 5 *For $1 \leq j \leq J$, $E\|\phi_{j,l}\|^4 = M(M+2)$, and $E\|\phi_{j,l}\|^8 = M(M+2)(M+4)(M+6)$.*

Proof of Lemma: Let $\phi_{j,n}$ be the column vector $[a_1, a_2, ..., a_M]^T$, Define the random variable $Z = \|\phi_{j,l}\|^2 = \sum_{q=1}^{M} a_q^2$. From the theory of random variables, we know that Z is chi-squared distributed with m degrees of freedom. Thus, $EZ^2 = M(M+2)$ and $EZ^4 = M(M+2)(M+4)(M+6)$, which proves the lemma.

Statistics of θ_n when $n \notin \Omega$

Assume without loss of generality that $\Omega = \{1, 2, \ldots, K\}$ for convenience of presentation. Let us compute the mean and variance of the test statistic θ_n for the case when $n \notin \Omega$. Consider one of these statistics by choosing $n = K + 1$.

Let $B = \langle y_j, \phi_{j,K+1} \rangle = \sum_{l=1}^{K} x_i(l) \langle \phi_{j,l}, \phi_{j,K+1} \rangle$. Clearly, the expectations of odd powers of B are zero, because $E(x_j(l)) = 0$ and $x_j(l)$ is independent of the other factors in each term of the summation. We will now compute EB^2 and EB^4. First, consider EB^2.

$$
\begin{aligned}
EB^2 &= E\left[\sum_{l=1}^{K} x_j(l)\langle \phi_{j,l}, \phi_{j,K+1}\rangle\right]^2 \\
&= E\left[\sum_{l=1}^{K} (x_j(l))^2 \left(\langle \phi_{j,l}, \phi_{j,K+1}\rangle\right)^2\right] + E\left[\sum_{q=1}^{K}\sum_{l=1,l\neq q}^{K} x_j(l)x_j(q)\langle \phi_{j,l}, \phi_{j,K+1}\rangle\langle \phi_{j,q}, \phi_{j,K+1}\rangle\right] \\
&= \sum_{l=1}^{K} E\left(x_j(l)\right)^2 E\left(\langle \phi_{j,l}, \phi_{j,K+1}\rangle\right)^2 + \\
&\quad \sum_{q=1}^{K}\sum_{l=1,l\neq q}^{K} E(x_j(l))E(x_j(q))E(\langle \phi_{j,l}, \phi_{j,K+1}\rangle\langle \phi_{j,q}, \phi_{j,K+1}\rangle) \\
&\quad \text{(because the terms are independent)} \\
&= \sum_{l=1}^{K} E\left(x_j(l)\right)^2 E\left(\langle \phi_{j,l}, \phi_{j,K+1}\rangle\right)^2 \qquad \text{(because } E(x_j(l)) = E(x_j(q)) = 0\text{)} \\
&= \sum_{l=1}^{K} \sigma^2 M \qquad \text{(from Lemma 1)} \\
&= MK\sigma^2.
\end{aligned}
$$

Next, consider EB^4.

$$
\begin{aligned}
EB^4 &= E\left[\sum_{l=1}^{K} x_i(l)\langle \phi_{j,l}, \phi_{j,K+1}\rangle\right]^4 \\
&= E\left[\sum_{l=1}^{K} (x_j(l))^4 \left(\langle \phi_{j,l}, \phi_{j,K+1}\rangle\right)^4\right] + \\
&\quad \binom{4}{2} E\left[\sum_{q=1}^{K}\sum_{l=1,l\neq q}^{K} (x_j(l))^2 (x_j(q))^2 (\langle \phi_{j,l}, \phi_{j,K+1}\rangle)^2 (\langle \phi_{j,q}, \phi_{j,K+1}\rangle)^2\right].
\end{aligned}
$$

The cross terms that involve $x_j(l)$, $x_j(q)$, $(x_j(l))^3$, $(x_j(q))^3$ factors have zero expectation, and hence not shown in the above equation. To explain the $\binom{4}{2}$ factor in the above expression, note that we have $\binom{4}{2}$ ways of obtaining the product of two squared factors when we expand EB^4. Thus,

$$\begin{aligned}
EB^4 &= \sum_{l=1}^{K} E\left(x_j(l)\right)^4 E\left(\langle \phi_{j,l}, \phi_{j,K+1}\rangle\right)^4 + \\
&\quad 6\sum_{q=1}^{K}\sum_{l=1, l\neq q}^{K} E(x_j(l))^2 E(x_j(q))^2 E(\langle \phi_{j,l}, \phi_{j,K+1}\rangle^2 \langle \phi_{j,q}, \phi_{j,K+1}\rangle^2) \\
&\quad \text{(because the terms are independent)}
\end{aligned}$$

Let us consider the two terms in the above equation separately. Simplifying the first term, we get

$$\begin{aligned}
\sum_{l=1}^{K} E\left(x_j(l)\right)^4 E\left(\langle \phi_{j,l}, \phi_{j,K+1}\rangle\right)^4 &= 3k\sigma^2 E\left(\langle \phi_{j,l}, \phi_{j,K+1}\rangle\right)^4 \\
&= 9K\sigma^4 M(M+2) \quad \text{(from Lemma 2)}.
\end{aligned}$$

The second term can be reduced to

$$6\sum_{q=1}^{K}\sum_{l=1, l\neq q}^{K} E(x_j(l))^2 E(x_j(q))^2 E(\langle \phi_{j,l}, \phi_{j,K+1}\rangle^2 \langle \phi_{j,q}, \phi_{j,K+1}\rangle^2)$$

$$\begin{aligned}
&= 6\frac{K(K-1)}{2}\sigma^4 M(M+2) \quad \text{(from Lemma 3)} \\
&= 3\sigma^4 K(K-1)M(M+2).
\end{aligned}$$

Summing the two terms, we get

$$\begin{aligned}
EB^4 &= 9K\sigma^4 M(M+2) + 3\sigma^4 K(K-1)M(M+2) \\
&= 3MK\sigma^4(M+2)(K+2).
\end{aligned}$$

Thus, we have

$$B = \langle y_j, \phi_{j,K+1}\rangle = \text{Moments}(0, MK\sigma^2, 0, 3MK\sigma^4(M+2)(K+2)).$$

Thus the first two moments for $\langle y_j, \phi_{j,K+1}\rangle^2$ are

$$\langle y_j, \phi_{j,K+1}\rangle^2 = \text{Moments}(MK\sigma^2, 3MK\sigma^4(M+2)(K+2)).$$

Writing in terms of cumulants,

$$\begin{aligned}
\langle y_j, \phi_{j,K+1}\rangle^2 &= \text{Cumulants}(MK\sigma^2, 3MK\sigma^4(M+2)(K+2) - M^2K^2\sigma^4) \\
&= \text{Cumulants}\left(MK\sigma^2, 2MK\sigma^4(MK+3K+3M+6)\right).
\end{aligned}$$

Summing J such independent random variables,

$$\sum_{j=1}^{J} \langle y_j, \phi_{j,K+1} \rangle^2 = \text{Cumulants}\left(MKJ\sigma^2, 2MKJ\sigma^4(MK + 3K + 3M + 6) \right).$$

Dividing by J,

$$\frac{1}{J} \sum_{j=1}^{J} \langle y_j, \phi_{j,K+1} \rangle^2 = \text{Cumulants}\left(MK\sigma^2, \frac{2MK\sigma^4}{J}(MK + 3K + 3M + 6) \right).$$

The above equation gives the mean and the variance of the test statistic θ_n when $n \notin \Omega$.

Statistics of θ_n when $n \in \Omega$

Again, we assume without loss of generality that $\Omega = \{1, 2, \ldots, K\}$ for ease of presentation. Let us compute the mean and variance of the test statistic θ_n for the case when $n \in \Omega$. Consider one of these statistics by choosing $n = 1$.

Let $G = \langle y_j, \phi_{j,1} \rangle = x_j(1) \|\phi_{j,1}\|^2 + \sum_{l=2}^{K} x_j(l) \langle \phi_{j,l}, \phi_{j,1} \rangle$. As before, the expectations of odd powers of G are zero, because of the leading $x_j(.)$ factor in each term. We will now compute EG^2 and EG^4. First, consider EG^2.

$$\begin{aligned}
EG^2 &= E\left[x_j(1) \|\phi_{j,1}\|^2 + \sum_{l=2}^{K} x_j(l) \langle \phi_{j,l}, \phi_{j,1} \rangle \right]^2 \\
&= E\left[(x_j(1))^2 \|\phi_{j,1}\|^4 \right] + E\left[\sum_{l=2}^{K} (x_j(l))^2 \langle \phi_{j,l}, \phi_{j,1} \rangle^2 \right] \\
&\qquad \text{(All other cross terms have zero expectation)} \\
&= E\left(x_j(1)\right)^2 E\|\phi_{j,1}\|^4 + \sum_{l=2}^{K} E\left(x_j(l)\right)^2 E\langle \phi_{j,l}, \phi_{j,1} \rangle^2 \quad \text{(by independence)} \\
&= \sigma^2 M(M+2) + (K-1)\sigma^2 M \quad \text{(from Lemmas 1 and 5)} \\
&= M(M+K+1)\sigma^2.
\end{aligned}$$

Next, consider EG^4.

$$\begin{aligned}
EG^4 &= E\left[x_j(1)\|\phi_{j,1}\|^2 + \sum_{l=2}^{K} x_j(l)\langle\phi_{j,l},\phi_{j,1}\rangle\right]^4 \\
&= E\left[x_j(1)\|\phi_{j,1}\|^2\right]^4 \\
&\quad + E\left[\sum_{l=2}^{K} x_j(l)\langle\phi_{j,l},\phi_{j,1}\rangle\right]^4 \\
&\quad + \binom{4}{2} E\left[\left(x_j(1)\|\phi_{j,1}\|^2\right)^2\left(\sum_{l=2}^{K} x_j(l)\langle\phi_{j,l},\phi_{j,1}\rangle\right)^2\right] \\
&\quad \text{(all other cross terms have zero expectation).}
\end{aligned}$$

We use the result from Lemma 5 to simplify the first term, and the result from the fourth moment of the statistic θ_n when $n \notin \Omega$ for the second term, to get

$$\begin{aligned}
EG^4 &= 3\sigma^4 M(M+2)(M+4)(M+6) + 3M(K-1)\sigma^4(M+2)(K+1) \\
&\quad + 6E\left[\left(x_j(1)\|\phi_{j,1}\|^2\right)^2\left(\sum_{l=2}^{K} x_j(l)\langle\phi_{j,l},\phi_{j,1}\rangle\right)^2\right]. \qquad (2)
\end{aligned}$$

The last term in the above equation can be written as

$$\begin{aligned}
E\left[\left(x_j(1)\|\phi_{j,1}\|^2\right)^2\left(\sum_{l=2}^{K} x_j(l)\langle\phi_{j,l},\phi_{j,1}\rangle\right)^2\right] &= E\left[(x_j(1))^2\,\|\phi_{j,1}\|^4\sum_{l=2}^{K}(x_j(l))^2\,\langle\phi_{j,l},\phi_{j,1}\rangle^2\right] \\
&\quad \text{(all other cross terms have zero expectation)} \\
&= \sigma^4 E\left[\sum_{l=2}^{K}\|\phi_{j,1}\|^4\langle\phi_{j,l},\phi_{j,1}\rangle^2\right] \\
&= \sigma^4(K-1)E\left[\|\phi_{j,1}\|^4\langle\phi_{j,2},\phi_{j,1}\rangle^2\right] \\
&= \sigma^4(K-1)M(M+2)(M+4) \\
&\quad \text{(using result from Lemma 4).}
\end{aligned}$$

Substituting this result in Equation 2, we get

$$\begin{aligned}
EG^4 &= 3\sigma^4 M(M+2)(M+4)(M+6) + 3M(K-1)\sigma^4(M+2)(K+1) \\
&\quad + 6(K-1)M(M+2)(M+4)\sigma^4 \\
&= 3M\sigma^4(M^3 + 10M^2 + 31M + MK^2 + 2M^2K + 12MK + 2K^2 + 16K + 30)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
G = \langle y_j,\phi_{j,1}\rangle &= \text{Moments}(0, M\sigma^2(M+K+1), 0, \\
&\quad 3m\sigma^4(M^3 + 10M^2 + 31M + MK^2 + 2M^2K + 12MK + 2K^2 + 16K + 30)).
\end{aligned}$$

Thus the first two moments of $\langle y_j, \phi_{j,1}\rangle^2$ are

$$\begin{aligned}\langle y_j, \phi_{j,1}\rangle^2 &= \text{Moments}(M\sigma^2(M+K+1),\\ &\quad 3M\sigma^4(M^3+10M^2+31M+MK^2+2M^2K+12MK+2K^2+16K+30)).\end{aligned}$$

In terms of cumulants,

$$\begin{aligned}\langle y_j, \phi_{j,1}\rangle^2 &= \text{Cumulants}(M\sigma^2(M+K+1),\\ &\quad 3M\sigma^4(M^3+10M^2+31M+MK^2+2M^2K+12MK+2K^2+16K+30)\\ &\qquad - M^2\sigma^4(M+K+1)^2)\\ &= \text{Cumulants}(M\sigma^2(M+K+1),\\ &\quad M\sigma^4(34MK+6K^2+28M^2+92M+48K+90+2M^3+2MK^2+4M^2K)).\end{aligned}$$

Summing J such random variables,

$$\begin{aligned}\sum_{j=1}^{J}\langle y_j, \phi_{j,1}\rangle^2 &= \text{Cumulants}(JM\sigma^2(M+K+1),\\ &\quad JM\sigma^4(34MK+6K^2+28M^2+92M+48K+90+2M^3+2MK^2+4M^2K)).\end{aligned}$$

Dividing by J,

$$\begin{aligned}\frac{1}{J}\sum_{j=1}^{J}\langle y_j, \phi_{j,1}\rangle^2 &= \text{Cumulants}(M\sigma^2(M+K+1),\\ &\quad \frac{M\sigma^4}{J}(34MK+6K^2+28M^2+92M+48K+90+2M^3+2MK^2+4M^2K)).\end{aligned}$$

The above equation gives the mean and the variance of the test statistic θ_n when $n \in \Omega$.

Proof of Theorem 2

The statistic θ_n is the mean of J independent random variables $\langle y_j, \phi_{j,n}\rangle$. For large J, we can invoke the central limit theorem [2–5] to argue that the distribution of θ_n is Gaussian with mean and variance as given in Theorem 1.

The one shot algorithm successfully recovers Ω if the following condition is satisfied: $[\max(\theta_n), n \notin \Omega] < [\min(\theta_n), n \in \Omega]$. To compute the probability that the above condition holds, we derive the equations that describe the distributions for the maximum and minimum, respectively, of θ_n when $n \notin \Omega$ and when $n \in \Omega$. Define $\theta_{max} \triangleq [\max(\theta_n), n \notin \Omega]$, and $\theta_{min} \triangleq [\min(\theta_n), n \in \Omega]$.

Let x be an arbitrary real number. For $n \notin \Omega$, the probability that the statistic θ_n is less than x is given by its cumulative distribution function (CDF):

$$Pr[\theta_n < x] = \frac{1}{2}\left(1 + erf\left(\frac{x - m_b}{\sigma_b\sqrt{2}}\right)\right).$$

Since the cardinality of the set Ω' is $N-K$, the probability that all the corresponding statistics θ_n, $n \notin \Omega$ are less than x is given by the CDF of θ_{max}:

$$Pr[\theta_{max} < x] = \frac{1}{2^{N-K}} \left(1 + erf\left(\frac{x - m_b}{\sigma_b\sqrt{2}}\right)\right)^{N-K}. \tag{3}$$

The above equation assumes that the statistics θ_n are independent. In reality, this assumption is not valid. However, we make this assumption in order to get an approximate result.

Using similar arguments, the probability that all the corresponding statistics θ_n, $n \in \Omega$ are *greater* than x is given by

$$Pr[min(\theta_n, n \in \Omega) > x] = Pr[\theta_{min} > x] = \frac{1}{2^K} \left(1 - erf\left(\frac{x - m_g}{\sigma_g\sqrt{2}}\right)\right)^K.$$

For a given x, the probability that θ_{max} lies between x and $x + dx$ can be computed using the probability density function (PDF) of θ_{max}. The PDF of θ_{max} in turn can be computed by differentiating its CDF as given by Eqauation 3. Thus,

$$\begin{aligned} Pr(\theta_{max} &\in [x, x+dx]) = \frac{d}{dx}\left[\frac{1}{2^{N-K}}\left(1 + erf\left(\frac{x - m_b}{\sigma_b\sqrt{2}}\right)\right)^{N-K}\right] dx \\ &= \frac{1}{2^{N-K-1}} \frac{(N-K)}{\sigma_b\sqrt{2\pi}} \exp\left[-\left(\frac{x - m_b}{\sigma_b\sqrt{2}}\right)^2\right] \left(1 + erf\left(\frac{x - m_b}{\sigma_b\sqrt{2}}\right)\right)^{N-K} dx. \end{aligned}$$

Thus the probability of successfully recovering Ω is given by

$$\begin{aligned} p_s &= \int_{x=-\infty}^{x=\infty} Pr(\theta_{max} \in [x, x+dx]).Pr(\theta_{min} > x) \\ &= \frac{1}{2^{N-1}} \frac{(N-K)}{\sigma_b\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[1 + erf\left(\frac{x - m_b}{\sigma_b\sqrt{2}}\right)\right]^{N-K-1} \left[1 - erf\left(\frac{x - m_g}{\sigma_g\sqrt{2}}\right)\right]^K \exp\left[-\left(\frac{x - m_b}{\sigma_b\sqrt{2}}\right)^2\right] dx. \end{aligned}$$

This proves Theorem 2.

Since we assumed independence between the statistics θ_n, the above is only an approximation. Figure 1 illustrates the approximation formula given by Theorem 2 by comparing with simulation results.

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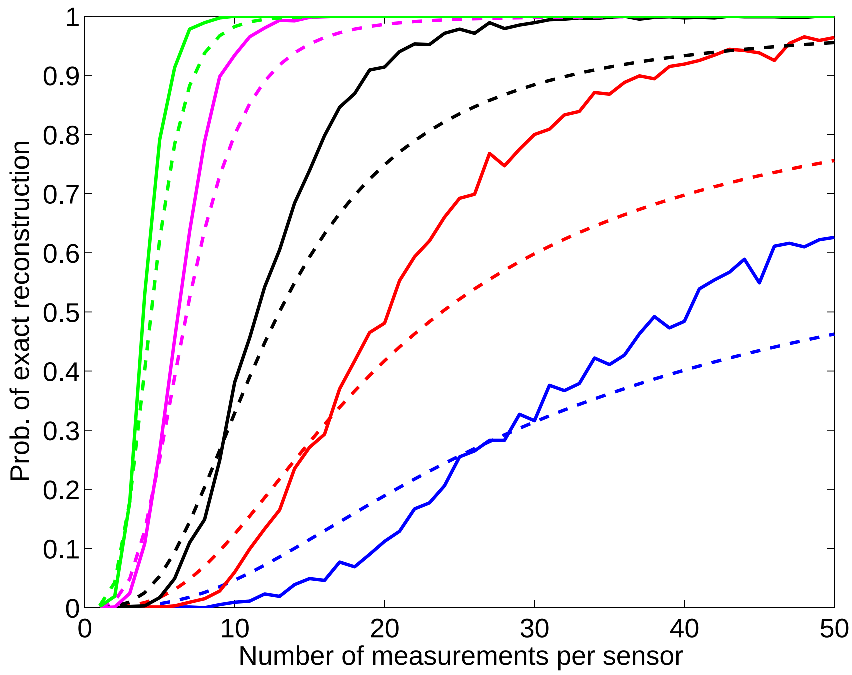

Figure 1: *Illustration of the approximate formula given by Theorem 2. The solid lines correspond to simulation results, and the dashed lines correspond to the formula given by Theorem 2. The blue curve correspond to* $J = 5$*, red* $J = 10$*, black* $J = 20$*, magenta* $J = 50$ *and green* $J = 100$*.*

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