Attenuation, dispersion, and anisotropy by multiple scattering of transmitted waves through distributions of scatterers

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Scattering of waves causes the coherent wave field to be attenuated and dispersed. These phenomena express the fact that the pulse loses coherence by which incoherent coda energy is created. First-order scattering theory violates the law of energy conservation and therefore cannot be used when the wave field is strongly distorted. An approximation is proposed which estimates the interaction between different scatterers by considering only multiple forward scattering interactions. Internal of a single scatterer all multiple interactions are maintained. This approximation can only be valid for the first part of the wave field and sufficiently weak scattering conditions. The heterogeneous medium can then be described as an effective medium that is a function of the scatterer density and forward scattering amplitude and the background medium. Simulations of the multiple scattering process with isotropic scatterers in two dimensions show that the discrepancies between the exact and approximate solution are small compared to the difference with the undisturbed wave field, even when the pulse is severely attenuated. Contrary to single scattering theory multiple scattering maintains the propagation of a stable and localized coherent wave. Apparently the nonlinear multiple scattering interactions cause a tendency for the coherent wave field to become insensitive to the specific scatterer distribution. © *1995 Acoustical Society of America.*

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INTRODUCTION

Heterogeneity of the constitutive parameters for seismic or acoustic waves can often be found at every spatial scale, e.g., in the Earth. Investigating this heterogeneity requires the understanding of its effect on seismic signals. The main effect of strong heterogeneity is an effective damping of the transmitted wave and the creation of incoherent energy (coda) through scattering.[1] Since no actual energy loss mechanism is needed to explain the damping of the coherent wave, the transfer of coherent energy to incoherent energy through scattering can be designated as *apparent* attenuation. The damping of the transmitted wave could conceivably be used to investigate the heterogeneity. The objective of this article is to make clear which parameters of the heterogeneity could, in principle, be extracted from the coherent part of the wave field. The strategy is to determine the effect of heterogeneity on the phase as well as the amplitude of the coherent wave. The forward problem for the coda will not be discussed.

Many existing theories on wave propagation through inhomogeneous media use a linear first-order scattering approximation known as the Born or Rayleigh–Gans approximation for describing the effect of heterogeneity on the seismic signal. In this approach the *scattered field* is taken *linearly proportional* with respect to the *perturbations in the constitutive parameters*. For example Snieder[2,3] adopted this linear approach to describe the effect of heterogeneity on the coherent wave as well as the coda for surface wave recordings of earthquakes. With the expressions obtained it was possible to invert measured data for a model of the lateral heterogeneity in the Earth's mantle under Europe and the Mediterranean.

Linear first-order scattering theory violates the fundamental law of energy conservation, hence the Born approximation cannot be used to analyze the amplitudes of seismic waves. Especially for high frequencies, when scattering conditions become stronger, attenuation must be described adequately and therefore *nonlinear interactions* of the *scattered field* with the *perturbations in the constitutive parameters* cannot be neglected.

Generally an inhomogeneous medium consists of a configuration of inhomogeneities. For a clear discussion a distinction should be made between nonlinear scattering interactions internal of a certain (volume) scatterer and nonlinear scattering interactions between different scatterers. We will refer to both processes as *multiple scattering* interaction. The total response of a scatterer including all multiple scattering can be given in terms of a scattering operator or scattering amplitude.[4] For a single scatterer the total attenuation and dispersion of the transmitted wave *including all multiple scattering interactions*, can be given in terms of the forward scattering amplitude and the optical theorem states that the total energy loss due to scattering and absorption can be related to the imaginary component of the forward scattering amplitude.

However for a configuration of scatterers the situation is more complicated. In the classical paper of Foldy[5] the mul-

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tiple scattering of a configuration of *isotropic* scatterers was treated by means of a self-consistent approach. The solution of the multiple scattering problem reduces to solving a large set of simultaneous linear algebraic equations. Because of the complexity of this problem and the absence of sufficiently large computers, Foldy resorted to a statistical approach of the problem. A configuration average was taken of the self-consistent equations in order to find an equation for the configurational average of the wave function. The configuration average of the wave function can also be designated as the mean wave field or the expectation value of the wave field. To obtain a closed equation for the mean wave field an approximation must be made for the configurational average of the external field incident on a certain scatterer. Foldy was forced to estimate this configurational external field by the mean total field which would exist at that point if the scatterer were not there. Later, more sophisticated approximations were made by Lax[6,7] with the postulate of an effective field or the quasicrystalline approximation (QCA). Waterman and Truell[8] extended the work of Foldy to anisotropic scatterers and correlations between scatterers. The main result was a backscattering correction for interactions between scatterers in terms of the backscattering amplitude. Similar statistical results were later obtained by means of the smoothing method.[9] Two important difficulties remain in all these approaches. First, it is the question which relation exists between the exciting field acting on a scatterer at a point, and the total field which would exist at that point if the scatterer were not there. No formal justifications exist for the postulated approximations. Second, there is the question of ergodicity; does the configurational average wave correspond with a certain *realization* or *measurement* of a transmitted wave for which the attenuation is accomplished by an averaging interference of scattered waves during propagation. Several papers address the problem of the interpretation of the configurational averaging procedure. Wu[10] has shown that differences of arrival time of specific realisations of the wave field lead to a reduced mean field amplitude which does not correspond with a true damping of the wave field. A correction for this effect is suggested by Sato[11] who first aligns these arrival times. Although more sophisticated statistical and computational methods exist such as the Maxwell Garnett approach,[12] the Coupled multipole method[13] and others[14,15] we will use yet another approach.

In this article we investigate the effect of forward scattering in case of a distribution of scatterers. The optical theorem can be used to describe the energy loss due to multiple scattering for a single isolated scatterer. For the inhomogeneous medium a model of isolated scatterers superposed on a homogeneous background medium is taken. By a *deterministic* approach the interference effects of the direct wave with the multiple scattered waves are evaluated. The interaction between different scatterers is estimated by using a multiple forward scattering approximation (MFSA). This approximation assumes that mainly interactions between scatterers in the forward direction contribute to the coherent part of the signal. For relatively low scatterer densities the main part of the energy related with backscattering interaction between scatterers simply arrive too late to contribute to the early part of the signal. Therefore the effect of backscattering on the early part of the signal is a higher order effect and shall be neglected. However, *when the path of propagation is sufficiently long*, relatively low scatterer densities can result in large attenuation of the coherent wave. The MFSA takes all multiple scattering interactions within each scatterer into account which leads to a specified value of the scattering coefficient for each scatterer. For an illustration of the scattering interactions considered in the MFSA see Fig. 1.

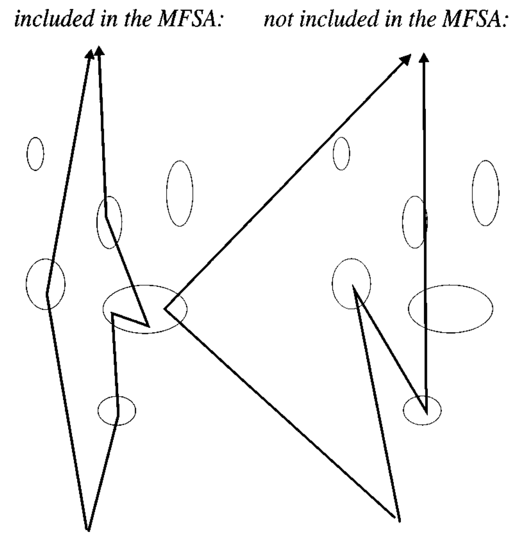

FIG. 1. Left: interactions included in the MFSA; forward, multiple forward, and backscattering inside a scatterer. Right: interactions not included; backscattering between scatterers and long delay interactions.

Instead of an equation for the mean field or configurational average an equation is found for the propagation of the coherent wave. Instead of an averaging over the statistical ensemble of random scatterers[5–8] the contributions of the scattered waves are integrated over the first Fresnel zone. Since all scattering contributions in the first Fresnel zone are by definition in phase the exact position of these scatterers is of minor importance and the integration over the Fresnel zone reduces to spatial averaging over the Fresnel zone. The problems related with the statistic approaches, such as the determination of the effective incident field, do not arise. However, because of ignoring backscattering between different scatterers, the resulting approximations are only valid for early times of the signal.

The MFSA suggests the heterogeneous medium to be described as a dynamic equivalent medium with attenuating, dispersive, and possibly anisotropic properties. The fact that the equivalent medium is frequency dependent is emphasized by designating the medium as dynamic. This equivalent medium is a function of the scatterer density, the forward scattering amplitude of the scatterers and the homogeneous background medium. In the MFSA the total effect of the scatterers is a pulse broadening and an effective damping of the coherent wave. The actual scatterer density used for calculating the MFSA is a weighted average over the Fresnel zone since scattering interactions related with scatterers outside the Fresnel zone arrive too late to contribute to the early part of the wave field.

The resulting expressions for the MFSA are to lowest order in scatterer density equivalent to statistical results of Foldy,[5] Lax[6,7] and Waterman and Truell,[8] although the interpretation is different. The MFSA describes the effect on the *early* part of the *coherent* wave in terms of a *spatial integration* over the first Fresnel zone. The statistical approach describes the effect on the *mean field* or configurational average in terms of the *ensemble average* of the scatterer density.

With the advent of large computers it is possible to invert the algebraic equations related with the multiple scattering problem as derived by Foldy[5] through the self-consistent approach. Therefore we are able to compare the MFSA with exact solutions including all multiple scattering effects. In fact this method is a simplified version of the coupled multipole method.[13] A comparison of the exact response with effective wave-number approximations has been done by Nelson,[16] who calculated the absorption coefficient. However, we emphasize the transient character of the multiple scattering process for a source pulse. Although our analysis is in the frequency domain, by neglecting backscattering we anticipate on its effect in the time domain after numerically inverting back to the time domain. Since the MFSA is postulated to be valid for early times of the signal, an adequate comparison with the exact response should also be made in the time domain. The exact response is therefore presented after inverting numerically back to the time domain. Numerical tests on wave propagation through distributions of isotropic point scatterers reveal that the discrepancies between the exact seismogram and the MFSA are small for the first part of the seismogram, compared to the difference with the wave that has traveled through the homogeneous background medium.

I. THEORY

For sake of simplicity the derivations are shown for two-dimensional scalar wave fields. The extension to three dimensions is straightforward. First, the scattering behavior of a single isolated scatterer is described. Second, the interaction between scatterers is estimated by a multiple forward scattering approximation. The total wave field can be separated into a wave field that is a solution to the homogeneous wave equation and a scattered wave field:

$$\Psi(\mathbf{r})=\Psi_0(\mathbf{r})+\Psi_S(\mathbf{r}). \tag{1}$$

Consider the response of a plane wave in the frequency domain with unit amplitude which impinges on a heterogeneity with bounded domain:

$$\Psi_0(\mathbf{r})=\exp(i\mathbf{k}_i\cdot\mathbf{r}). \tag{2}$$

The vector $\mathbf{k}$ defines the direction of propagation and the wavelength for the homogeneous background medium. We suppose the medium is instantaneously reacting and therefore the velocity c_0 is independent of frequency:

$$|\mathbf{k}_i|=k_0=\frac{2\pi}{\lambda}=\frac{\omega}{c_0}. \tag{3}$$

The scattered wave field in the far field due to the incident wave field defined in Eq. (2) is given in terms of a scattering amplitude:[4]

$$\Psi_S(\mathbf{r},\mathbf{r}') \underset{k_0|\mathbf{r}-\mathbf{r}'|\gg 1}{=} \Psi_0(\mathbf{r},\mathbf{r}')+G^{(0)}(\mathbf{r},\mathbf{r}') \times A(\mathbf{k}_s,\mathbf{k}_i)\Psi_0(\mathbf{r}'). \tag{4}$$

The scattering amplitude $A(\mathbf{k}_s,\mathbf{k}_i)$ is a function of the incident wave vector and the scattered wave vector $\mathbf{k}_s$. For the two-dimensional scalar wave equation the Green's function is given by

$$G^{(0)}(\mathbf{r},\mathbf{r}')=-\frac{i}{4}H_0^{(1)}(k_0|\mathbf{r}-\mathbf{r}'|) \cong \frac{\exp(i(k_0|\mathbf{r}-\mathbf{r}'|-3\pi/4))}{4\sqrt{\pi/2}\sqrt{k_0|\mathbf{r}-\mathbf{r}'|}}, \quad k_0|\mathbf{r}-\mathbf{r}'|\gg 1. \tag{5}$$

The function $H_0^{(1)}$ is the Hankel function of the first kind of zeroth order.

Generally the energy loss of the incident wave field passing the heterogeneity is caused by scattering and absorption. The total energy loss for unit incident wave field [Eq. (2)] is called the total cross-section $\Omega_{\mathrm{TOT}}(\mathbf{k}_i)$. The classical optical theorem states that the total scattering cross section can be related to the imaginary component of the forward scattering amplitude. In the case of two dimensions:

$$\Omega_{\mathrm{TOT}}(\mathbf{k}_i)=-\frac{\mathrm{Im}\,A(\mathbf{k}_i,\mathbf{k}_i)}{k_0}. \tag{6}$$

The optical theorem can for example be derived by using a stationary phase evaluation of the scattering integral in the far field.[17]

The description of the attenuation of the transmitted wave is extended to the case of an assemblage of isolated scatterers by using the optical theorem. Only scatterers within the Fresnel zone contribute to the early part of the transmitted wave field. By definition these scatterers radiate in phase with the background wave field, which means that the precise location of the scatterer is of minor importance. The discrete distribution of the scatterers in the Fresnel zone can therefore be replaced by a smooth scatterer density ν. The effect of the assemblage of scatterers is evaluated by again using a stationary phase approximation and the optical theorem. The actual stationary phase approximation is *not* valid for distributions of scatterers. This approximation however is only used to evaluate the contribution of the smooth scatterer distribution in the Fresnel zone that contributes to the early part of the wave field.

The amplitude distortion of a plane wave impinging perpendicularly on a very thin layer (thickness δ) of scatterers is determined first. In this layer the interactions between different scatterers are ignored. Later multiple interactions are incorporated by extending the layer to arbitrary thickness. This procedure is similar to the method presented by Fermi.[18] The definition of the geometric variables is shown in Fig. 2. The total wave field is calculated in the far field for large x:

$$\Psi(x,0)=T(\delta)\exp(ik_0x)$$
$$\underset{k_0|\mathbf{r}-\mathbf{r}'|\gg 1}{=}\exp(ik_0x)+\int_0^\delta\int_{-\infty}^{\infty}\nu A(\mathbf{k}_s,\mathbf{k}_i)\times G^{(0)}(x;x',y')\exp(ik_0x')dy'\,dx'. \tag{7}$$

In this expression $T(\delta)$ is the transmission coefficient for a layer of thickness δ. The argument of the Green's function and also the scattering amplitude in the far field can for large values of x be simplified:

$$|\mathbf{r}-\mathbf{r}'|=\sqrt{(x-x')^2+y'^2}\rightarrow(x-x')+\frac{y'^2}{2(x-x')},$$
$$A(\mathbf{k}_s,\mathbf{k}_i)\rightarrow A(\mathbf{k}_i,\mathbf{k}_i). \tag{8}$$

With this far-field approximation Eq. (7) becomes:

$$\Psi(x,0)\cong\exp(ik_0x)\left(1+\int_0^\delta\int_{-\infty}^{\infty}\frac{\nu A(\mathbf{k}_i,\mathbf{k}_i)\exp[i(k_0y'^2/2(x-x')-3\pi/4)]}{4\sqrt{\pi/2}\sqrt{k_0(x-x')}}\,dy'\,dx'\right). \tag{9}$$

Equation (9) can be evaluated with the method of stationary phase[19] resulting in

$$T(\delta)=\left(1-i\,\frac{\nu\delta}{2k_0}\,A(\mathbf{k}_i,\mathbf{k}_i)\right). \tag{10}$$

Equation (10) represents the transmission coefficient for a thin layer. It should be understood that because of the stationary phase approximation, only the scatterers in the first Fresnel zone contribute to this expression. A simplified version of the invariant imbedding technique[20,21] is used to calculate the transmission coefficient of a layer of arbitrary thickness. The total medium is composed by subsequently adding thin layers. It is assumed that the scatterer density is sufficiently small so that multiple backscattering between different scatterers can be ignored. This approximation is based on the argument of arrival time of the bulk of backscattered energy. Although neglecting backscattering is not valid in the frequency we anticipate on its effect after inversion to the time domain. The increase of the transmission coefficient due to adding a thin layer of scatterers is calculated using Eq. (10). By doing so, we include multiple forward scattering interactions and neglect backscattering between different scatterers:

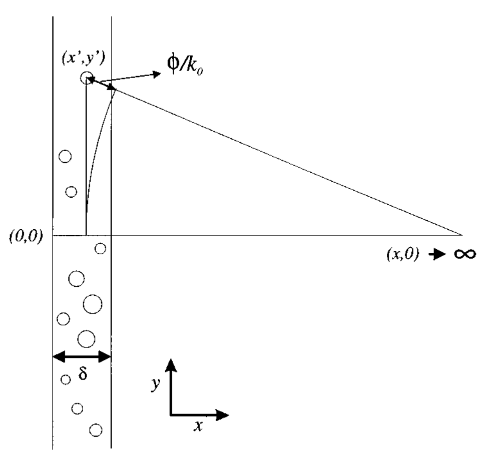

FIG. 2. Explanations of geometrical variables used for calculation of the transmission coefficient of a thin layer of scatterers.

$$T(\Delta+\delta)=T(\Delta)T(\delta)=T(\Delta)\left(1-i\,\frac{\nu\delta}{2k_0}\,A(\mathbf{k}_i,\mathbf{k}_i)\right). \tag{11}$$

Equation (11) is equivalent to the differential equation:

$$\lim_{\delta\to 0}\frac{T(\Delta+\delta)-T(\Delta)}{\delta}=\frac{\partial T(\Delta)}{\partial\Delta}=-iT(\Delta)\,\frac{\nu A(\mathbf{k}_i,\mathbf{k}_i)}{2k_0}. \tag{12}$$

With the boundary condition that in absence of scatterers the plane wave is undisturbed $[T(0)=1]$, the solution becomes:

$$T(\Delta)=\exp\left(i\int_0^\Delta\frac{\nu}{2k_0}\,A(\mathbf{k}_i,\mathbf{k}_i)d\xi\right)$$
$$=\exp\left(-i\,\frac{\langle\nu A(\mathbf{k}_i,\mathbf{k}_i)\rangle}{2k_0}\,\Delta\right). \tag{13}$$

As noted earlier, only the scatterers in the first Fresnel zone contribute to the stationary phase integral in Eq. (9). The additional integration over ξ in the first line of Eq. (13) implies that only the average of $\nu A(\mathbf{k}_i,\mathbf{k}_i)$ over the first Fresnel zone is of relevance for the propagation of the coherent wave. To obtain the link between the spatially averaged scatterer density and the discrete scatterer distribution, the averaged scatterer density can for a given configuration of scatterers be estimated with a weighted sum:

$$\langle\nu A(\mathbf{k}_i,\mathbf{k}_i)\rangle=\frac{\Sigma_j w(\mathbf{r}_j)A(\mathbf{k}_i,\mathbf{k}_i)_j}{\int_{\mathbb{R}^2}w(\mathbf{r})d\mathbf{r}}, \tag{14}$$

where w is the positive weighting function and the $\mathbf{r}_j$ are the coordinates of the scatterers. The variability of the different scatterer distributions for different source receiver pairs is modeled in the MFSA by using the averaging in Eq. (14) for every individual wave path. In the special case of a homogeneous scatterer density a plane wave effectively propagates with a complex wave number k_{eff}:

$$k_{\text{eff}}=k_0\left(1-\frac{\nu A(\mathbf{k}_i,\mathbf{k}_i)}{2k_0^2}\right). \tag{15}$$

To leading order the corresponding velocity is given by

$$c_{\mathrm{eff}} \cong c_0\left(1+\frac{\nu\ \mathrm{Re}(A(\mathbf{k}_i,\mathbf{k}_i))}{2k_0^2}\right). \quad (16)$$

Bohren and Singham[22] have shown that the forward scattering amplitude is relative insensitive to the shape of the scatterer. Assuming the MFSA is a valid approximation, the attenuation and dispersion of the transmitted wave is therefore relatively insensitive to the shapes of the scatterers. Foldy's[5] result for the effective wave number was [with similar convention for the definition of the scatterer amplitude as in Eq. (4)]:

$$\begin{aligned} k_{\mathrm{eff}} &= k_0\sqrt{1-\frac{\langle \nu A(\mathbf{k}_i,\mathbf{k}_i)\rangle}{k_0^2}} \\ &= k_0\left(1-\frac{\langle \nu A(\mathbf{k}_i,\mathbf{k}_i)\rangle}{2k_0^2}\right)+O\left(\frac{\langle \nu A(\mathbf{k}_i,\mathbf{k}_i)\rangle}{k_0^2}\right)^2. \end{aligned} \quad (17)$$

From Eq. (17) it can be seen that to lowest order in the scatterer density the effective wave numbers of the MFSA [Eq. (15)] and Foldy [Eq. (17)] are equal. Note that the configurational average scatterer density used by Foldy is in the MFSA replaced by the average scatterer density over the first Fresnel zone.

The effective wave number derived by Waterman and Truell[8] is (again with similar definition of the scatterer amplitude):

$$\begin{aligned} k_{\mathrm{eff}} &= \sqrt{\left(k_0^2-\frac{\langle \nu A(\mathbf{k}_i,\mathbf{k}_i)\rangle}{k_0^2}\right)^2-\left(\frac{\langle \nu A(\mathbf{k}_i,-\mathbf{k}_i)\rangle}{k_0^2}\right)^2} \\ &= k_0\left(1-\frac{\langle \nu A(\mathbf{k}_i,\mathbf{k}_i)\rangle}{2k_0^2}\right)+O\left(\frac{\langle \nu A(\mathbf{k}_i,\pm\mathbf{k}_i)\rangle}{k_0^2}\right)^2, \end{aligned} \quad (18)$$

where it is assumed that the backscattering amplitude $A(\mathbf{k}_i,-\mathbf{k}_i)$ is in the order of the forward scattering amplitude. This is, in fact, an overestimation of $A(\mathbf{k}_i,-\mathbf{k}_i)$ because for scatterers with a finite size, the forward scattering is much stronger than the backscattering. The effective wave number [Eq. (15)] agrees to leading order with the effective wave number derived in Refs. 5–8, as can be seen in Eqs. (17) and (18).

Our postulate is that the effective wave number in the MFSA is a valid approximation if the relative change of this wave number is small:

$$\left|\frac{k_{\mathrm{eff}}-k_0}{k_0}\right| = \frac{\langle\nu\rangle A(\mathbf{k}_i,\mathbf{k}_i)}{2k_0^2} \ll 1. \quad (19)$$

However, this small correction to the homogeneous wave number can result in *large attenuation effects*. The amplitude $|T|$ of the transmitted plane wave and its phase shift $\Delta\phi$ are

$$\begin{aligned} |T| &= \exp\left(\frac{\langle \nu\ \mathrm{Im}\ A(\mathbf{k}_i,\mathbf{k}_i)\rangle}{2k_0}\ \Delta\right), \\ \Delta\phi &= -\frac{\langle \nu\ \mathrm{Re}\ A(\mathbf{k}_i,\mathbf{k}_i)\rangle}{2k_0}\ \Delta. \end{aligned} \quad (20)$$

Since the effective wave number enters the exponent multiplied with the path length Δ the amplitude attenuation and phase shift are not necessarily small. The MFSA can be expected to be an approximation only for the early part of the wave field because then the main part of the backscattering interactions can be neglected since the multiply backscattered waves simply arrive too late. For small scatterer densities backscattering interactions can be neglected because its effect will be of a higher order in the scatterer density. Numerical tests show that, for the later part of the signal, nonforward scattering is crucial for the correct description of the multiple scattering process.

Assuming that the MFSA is a reasonable approximation, the heterogeneous medium can for the coherent wave field be described as an effective homogeneous replacement medium. Three main effects of the heterogeneity can be seen. First, the coherent wave is attenuated because of the requirement of energy conservation. Second, the corrected wave number is a function of the forward scattering amplitude, the scatterer density, and the homogeneous wave number, all of which can be frequency dependent. The effective wave number therefore also describes dispersion of the coherent wave field. Third, the effective wave number also contains an anisotropic component. The cause of anisotropy in the MFSA description of the effective velocity is twofold. A certain preferred orientation of the scatterers results in a directionally dependent forward scattering amplitude and therefore in a directionally dependent effective wave number. A different average distance between scatterers results in different weighted averaged scatterer densities over the zone of stationary phase and therefore in effective anisotropy. However, the MFSA does not contain a backscattering correction for correlations between scatterers.

To get insight in the accuracy of the approximation, the MFSA is compared with the exact solution for distributions of isotropic point scatterers.

II. NUMERICAL MODELING OF THE WAVE FIELD FOR ISOTROPIC SCATTERERS

The validity of the MFSA can only be assessed in case a comparison can be made with an exact solution. However, for distributions of a limited number of isotropic point scatterers the solution of the multiple scattering process can be reduced to a linear system of equations by means of the self-consistent approach as in Foldy.[5] For every frequency component a square matrix with rank equal to the number of scatterers has to be inverted. For details see the Appendix.

The approach described above is taken for calculating the response for isotropic point scatterers in two dimensions. These scatterers radiate equally in every direction and essentially reduce to line scatterers in the three-dimensional world. The wave field scattered by a single point scatterer at location $\overline{\mathbf{r}'}$ due to an incident wave field $\Psi_0(\mathbf{r})$ is given by

$$\Psi_S(\mathbf{r}) = G^{(0)}(\mathbf{r},\mathbf{r}')A\Psi_0(\mathbf{r}'). \quad (21)$$

The scattering amplitude A is independent of the scattering angle. We assume that energy loss of the incident wave is only due to scattering, anelastic attenuation is not taken into account. Because of the optical theorem a relation exists between the real and imaginary component of the scattering amplitude. The total cross section Ω_{TOT} related with the imaginary component of the (forward) scattering amplitude [Eq. (6)] is equal to the scattering cross-section Ω_S, or the

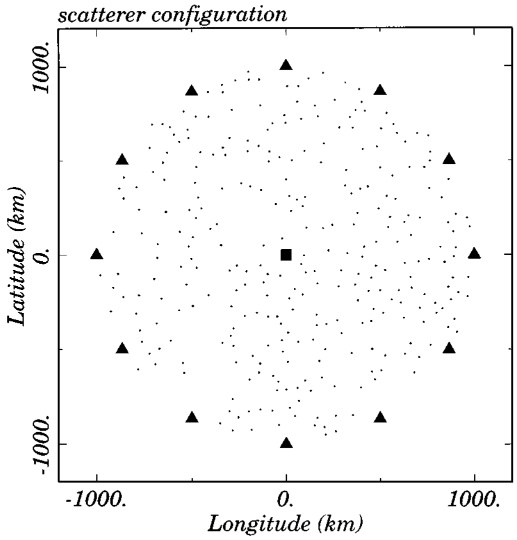

FIG. 3. Configuration of the source (square), receivers (triangles), and scatterers (dots).

total energy flow in all directions for unit incident plane wave:

$$\Omega_{\mathrm{TOT}}=-\frac{\mathrm{Im}\,A}{k_0},$$
$$\Omega_S=\int\int_{\partial V}|\Psi_S(\mathbf{r})|^2\,d^2\mathbf{r}=\frac{1}{8k_0\pi}\int_0^{2\pi}|A|^2\,d\theta=\frac{|A|^2}{4k_0}, \tag{22}$$

where Eq. (5) and Eq. (21) have been used. Equating the total cross section and the scattering cross section leads to

$$\mathrm{Re}\,A=\pm\sqrt{-\mathrm{Im}\,A(4+\mathrm{Im}\,A)}. \tag{23}$$

The positive and negative sign correspond with a phase advance and delay, respectively. Equation (23) imposes a constraint on the value of the imaginary component of the scattering amplitude and consequently on the total cross section:

$$-4\leqslant \mathrm{Im}\,A\leqslant 0 \quad \text{or} \quad 0\leqslant\Omega_{\mathrm{TOT}}\leqslant 4/k_0. \tag{24}$$

This means that for the rather idealized isotropic point scatterers the strength of a single scatterer is bounded by the requirement that energy is conserved.

TABLE I. Values of parameters used for calculating the MIPS, MFSA, BORN, and BACK seismograms.

Quantity	Value	Explanation
c_0	4.0 km/s	Background velocity
Δ	1000 km	Source/receiver distance
γ	2.7	Strength of scatterers
f_{cut}	0.05 Hz	Limit frequency dependency of scatterer amplitude
$f_{\min}$	0.01 Hz	Source frequency band taper
$f_{\max}$	0.1 Hz	Source frequency band taper
N	300	Number of scatterers
dt	2.5 s	Sample time
n	1024	Number of samples

To produce realistic looking seismograms the scattering amplitude is defined to have a frequency dependence proportional to f^2. This is the frequency dependence usually obtained from single scattering theory.[1,2] For high frequencies the scattering amplitude is in this example defined to be independent of frequency. Using Eq. (23) the scattering amplitude can be defined in terms of its imaginary part:

$$\mathrm{Im}\,A=\begin{cases}-\gamma(f^4/f_{\mathrm{cut}}^4), & f\leqslant f_{\mathrm{cut}}\\ -\gamma, & f\geqslant f_{\mathrm{cut}}\end{cases} \qquad \gamma<4. \tag{25}$$

The imaginary component defines by virtue of Eq. (23), apart from the sign, also the real part of the scattering amplitude.

The source spectrum of the incident wave in the numerical experiment is given by

$$\hat{f}=\frac{f-f_{\min}}{f_{\max}-f_{\min}},$$
$$S(\hat{f})=\begin{cases}\frac{1}{2}(1-\cos(2\pi\hat{f})), & 0<\hat{f}<1\\ 0, & \text{otherwise}\end{cases}. \tag{26}$$

Note that Eq. (15) is derived for a plane incoming wave rather than a point source. However, such a plane geometry differs from a point source situation predominantly in geometrical spreading. The attenuation and dispersion of the direct wave is caused by phase interference of the incident wave with the scattered waves. Similar as in Waterman and Truell[8] (see the Appendix) the transmission coefficient of a finite layer [Eq. (13)] is generalized to obtain:

$$\begin{aligned}\Psi(\Delta)&=G^{(0)}(\Delta)T(\Delta)\\ &\cong\frac{\exp(i(k_0\Delta-3\pi/4))}{4\sqrt{\pi/2}\sqrt{k_0\Delta}}\exp\left(-i\,\frac{\langle\nu A(\mathbf{k}_i,\mathbf{k}_i)\rangle}{2k_0}\,\Delta\right)\\ &=\frac{\exp(i(k_{\mathrm{eff}}\Delta-3\pi/4))}{4\sqrt{\pi/2}\sqrt{k_0\Delta}}.\end{aligned} \tag{27}$$

The weighting function defined in Eq. (14) must be defined. We do not claim to present the optimal weighting function but our postulate is that the actual sensitivity to the scatterer distribution is weak because all scatterers in the Fresnel zone radiate in phase. The weighting function will be mainly a function of the detour of the scattered wave. The detour is defined as the difference of the scattered path and the direct path:

$$\Delta s(\mathbf{r})=|\mathbf{r}-\mathbf{r}_s|+|\mathbf{r}_r-\mathbf{r}|-|\mathbf{r}_r-\mathbf{r}_s|. \tag{28}$$

The coordinates of the source and receiver are denoted as $\mathbf{r}_s$ and $\mathbf{r}_r$, respectively. The employed weighting function in the numerical experiment has the following form:

$$\Delta\hat{s}=\frac{k_0}{\pi/2}\,\Delta s(\mathbf{r}),$$
$$w(\Delta\hat{s})=\begin{cases}1+\cos(\pi\Delta\hat{s}), & \Delta\hat{s}<1\\ 0, & \text{otherwise}\end{cases}. \tag{29}$$

The normalized detour $\Delta\hat{s}$ defines a cosine tapered weighting function over the (frequency-dependent) Fresnel zone. The

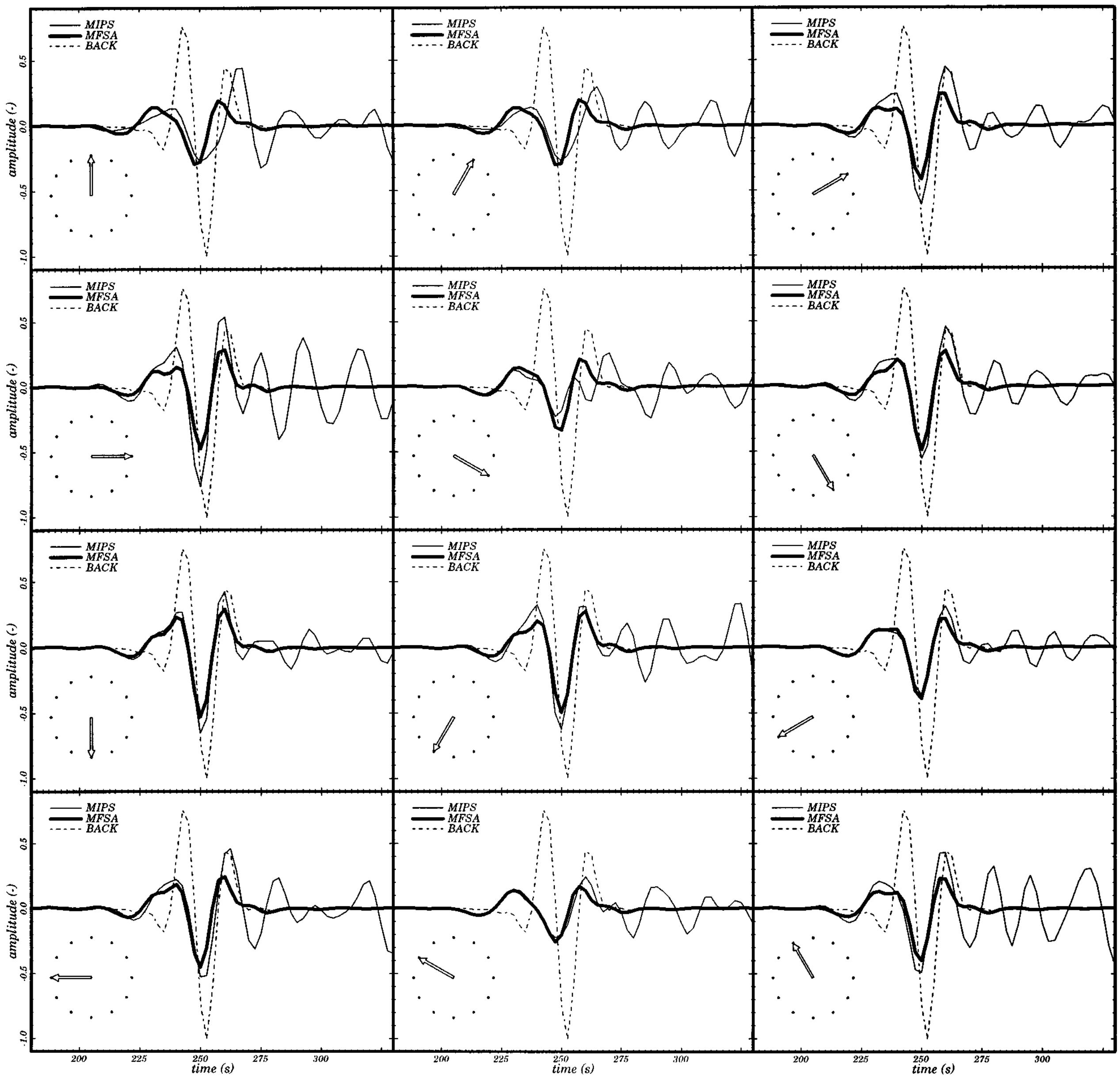

FIG. 4. The multiple isotropic point scattering solution (MIPS, thin solid line), The multiple forward scattering approximation (MFSA, thick solid line) and undisturbed wave field (BACK, dotted line). Scatterers have positive real part of scattering amplitude. The receiver location is indicated by an arrow in the lower left corner.

source and receiver act as focal point for the ellipses, that define the contours of constant value for the weighting function.

III. ACCURACY OF THE MFSA

For twelve receivers located concentrically around a single source a comparison was made between the multiple isotropic point scattering seismograms (MIPS, see the Appendix) and the multiple forward scattering approximation (MFSA) relative to the homogeneous background medium seismogram (BACK). The MIPS seismograms will be considered to be the exact response of the medium with the scatterers. The configuration of the source, receivers, and scatterers is shown in Fig. 3. The unit distance is a kilometre. The employed parameters for the background medium, source and scatterer characteristics are listed in Table I. The parameters chosen are exemplary for surface waves triggered by earthquakes traveling in the Earth's mantle. The scatterer densities and the related corrections of the wave number in the experiment are small (at the maximum around 20% for f_{cut}, which is around the central frequency).

In the first example the sign of the real part of the scattering amplitude is chosen positive. The wavefront obtains a phase advance caused by the positive velocity anomalies. The resulting seismograms for the twelve receivers are shown in Fig. 4. Although the relative perturbation of the wave number is small the direct wave is strongly attenuated, because of the length of the path of propagation. This example of multiple scattering is not in the weak scattering regime; the scattered wave arriving at the tail of the direct

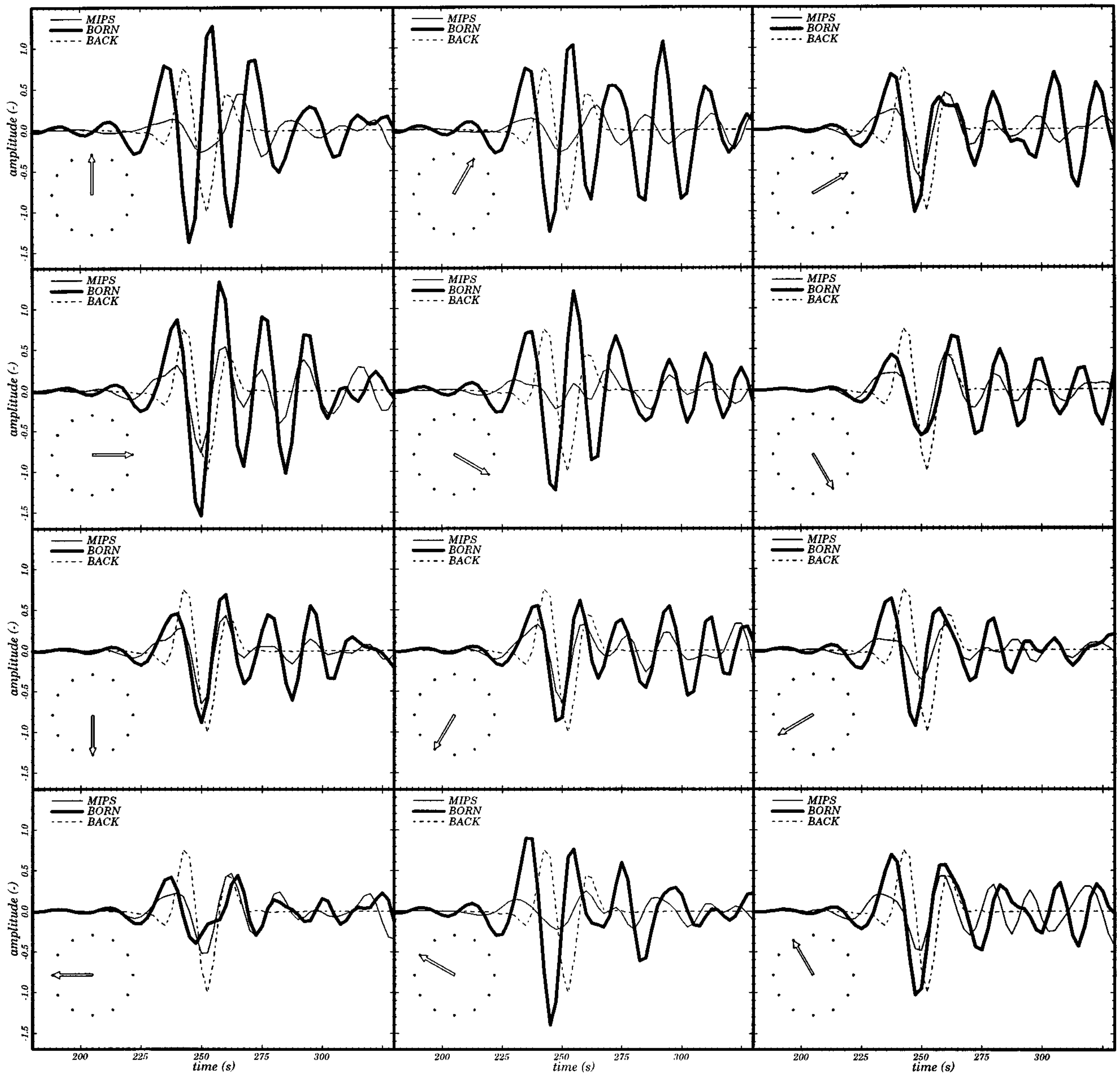

FIG. 5. The multiple isotropic point scattering solution (MIPS, thin solid line), The single scatterer approximation (BORN, thick solid line) and undisturbed wavefield (BACK, dotted line). The receiver location is indicated by an arrow in the lower left corner. BORN seems to be an unstable approximation.

wave ($t>275$s.) has a magnitude comparable to the direct arriving wave field. Note that the amplitude of the direct wave field shows considerable variability between different receivers due to variability in the scatterer distributions. This variability is handled well by the MFSA. The MFSA accounts for this variability because it uses the scatterer density averaged over the first Fresnel zone for each individual path. With the configurational average scatterer density of statistical methods this aspect is more difficult to implement.

The discrepancies between the MFSA and the MIPS seismograms are small for the early part of seismogram compared to the difference of the MIPS with the undisturbed (BACK) seismogram. Apparently the inhomogeneous medium behaves like an effective homogeneous medium for the early or coherent part of the wave field. The scatterer distributions exhibit considerable fluctuation in the scatterer density. As expected the MFSA does not describe the later part of the seismogram. In this regime backscattering between scatterers and wide-angle scattering dominates.

In order to test the accuracy of single scattering and to investigate whether the strong attenuation is really due to multiple scattering interactions, the single scattering approximations (BORN) for the same configuration as in the first example were calculated, using Eq. (A2) of the Appendix. The single scattering approximations are denoted as BORN seismograms because this approximation is intimately related with the BORN approximation in the sense that the wave field at the scatterer is estimated by the incident wave field.

In Fig. 5 the single isotropic scattering seismograms (BORN) are plotted with thick lines instead of the MFSA.

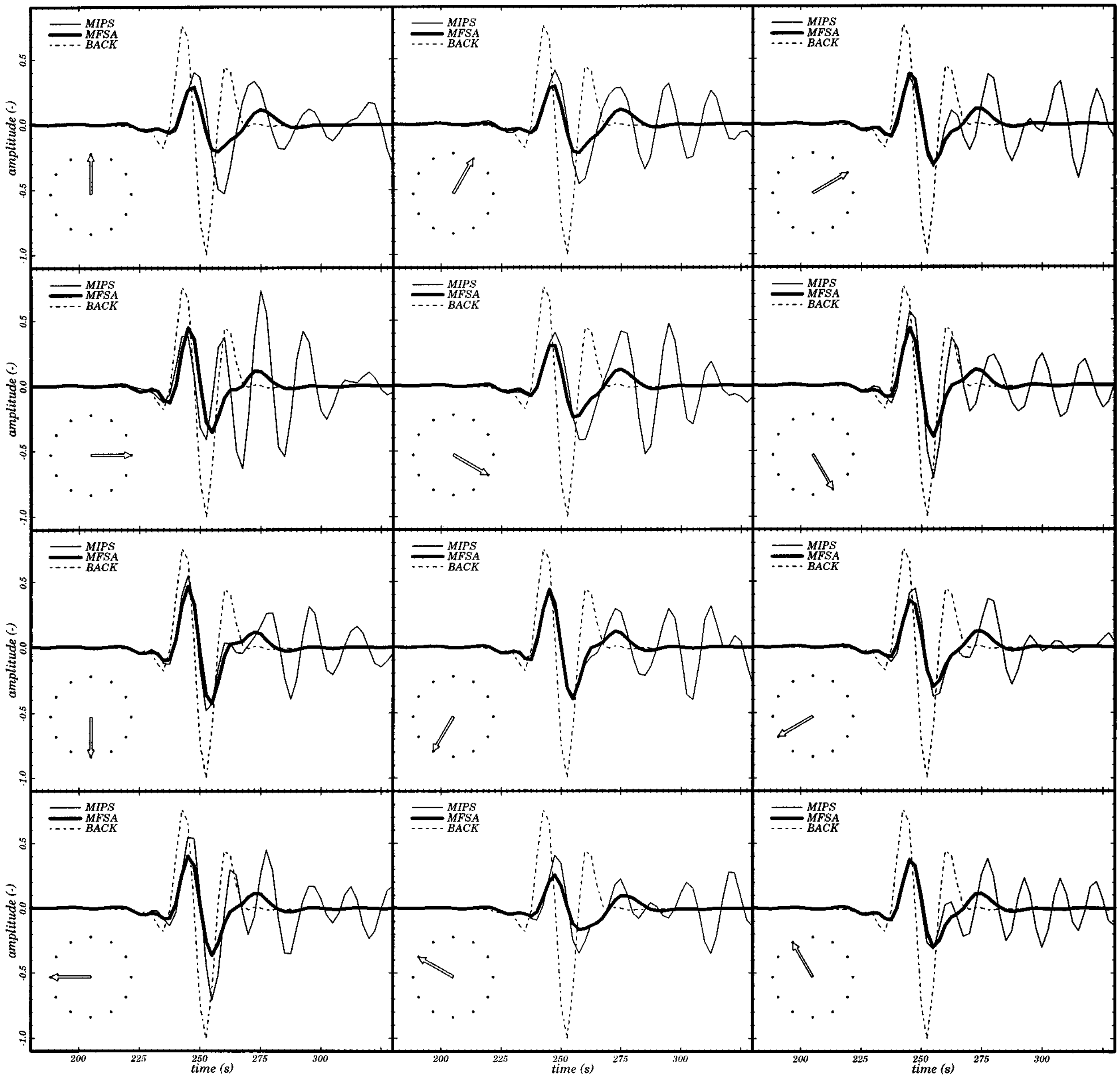

FIG. 6. The multiple isotropic point scattering solution (MIPS, thin solid line), The multiple forward scattering approximation (MFSA, thick solid line) and undisturbed wave field (BACK, dotted line). Scatterers have *negative* real part of scattering amplitude. The receiver location is indicated by an arrow in the lower left corner.

Note that the vertical scale in the Figs. 4 and 5 is different. The MFSA (Fig. 4) gives a much better estimate than the BORN seismograms (Fig. 5). The BORN seismograms are very unstable or unrealistically sensitive to the specific scatterer distribution. The MIPS and the MFSA solution maintain the existence of a localized coherent wave. This is not seen in case of the BORN approximation, because the signal consists of long reverberations. The paradoxical conclusion is obtained that by including more complex scattering interactions a more simple wave field results.

Apparently the nonlinear interaction between scatterers stabilizes the waveform distortion. The MFSA describes this stabilization effect adequately.

In Fig. 4 the scatterers induce a negative time shift to the transmitted wave field. When the sign of the real part of the scattering amplitude in Eq. (23) reverses, the scatterers induce a positive time lag to the scattered wave field. The resulting positive time shift can clearly be seen in Fig. 6. Apparently the MFSA is able to handle both the positive and negative time shifts adequately.

For a number of different configurations the MFSA was compared to the MIPS seismograms. These configurations included inhomogeneous distributions as well as different strengths for the scattering amplitude and dispersive background media. Similar results were obtained except for very strong attenuation and also for very low scatterer densities and strong scatterers. In the first case the effect of backscattering becomes important also for the early part of the wave field. The generalized primary almost disappears. In the second case a limited number of scatterers in the Fresnel zone do not accomplish the apparent homogeneity that is needed for the validity of the MFSA.

IV. CONCLUSIONS

A deterministic approximation is presented that accounts for the attenuation, dispersion, and anisotropy caused by multiple scattering of the early part of the wave field by distributions of scatterers. Backscattering between different scatterers is ignored, therefore this approximation is only valid for the early or coherent part of the wave field and not too dense scatterer distributions. The complications related with a statistical approach namely the closure approximations or assumptions (e.g., QCA) and the question of ergodicity are avoided.

Tests with isotropic point scatterers show that the discrepancies between the exact and the multiple forward scattering approximation seismograms are small compared to the difference with the seismogram of the undisturbed wave field for the coherent first arriving wave. Apparently, the inhomogeneous medium then approximately behaves like an effective homogeneous medium. The multiple scattering creates a tendency of the early part of the wave field to become insensitive to the specific scatterer distribution and to maintain a localized coherent wave field. This behavior is not seen for single isotropic point scattering. The variability of the attenuation and dispersion for different receivers is retained in the MFSA. Therefore a main advantage of this approximation is its clear interpretation of the scatterer density in terms of a spatial averaged scatterer density over the Fresnel zone.

The multiple forward scattering approximation neglects backscattering between different scatterers. The isotropic point scattering normally overestimate the effect of backscattering compared to nonisotropic scatterers, because in general the finite size of the scatterers leads to an enhanced forward scattering, the Mie effect.[23] We expect the multiple forward scattering approximation therefore to behave more accurate for nonisotropic scatterers than for point scatterers, because the neglected backscattering between scatterers is usually weaker for scatterers with a finite size.

In the MFSA the corrected effective medium can be described by the forward scattering amplitude, the scatterer density averaged over the first Fresnel zone and the background medium. Using measurements of attenuation and dispersion of the coherent wave field only an estimate of the spatial average of these parameters over the Fresnel zone can be obtained. The specific scatterer distribution within the Fresnel zone cannot be extracted from a single registration of the coherent part of the wave field. The effective wave number is only sensitive to the forward scattering amplitude and thus weakly sensitive to the shape of the scatterers. Only by combining crossing wave paths or using the incoherent energy a more detailed description of the spatial distribution of the scatterers can possibly be obtained.

ACKNOWLEDGMENTS

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APPENDIX: THE SOLUTION OF THE MULTIPLE SCATTERING PROCESS FOR DISTRIBUTIONS OF ISOTROPIC POINT SCATTERERS

When the scattering interaction is limited to point scatterers, the multiple scattering process can be reduced to a linear system of equations that can be solved numerically.[5] The wave field is described in the frequency domain and is written as the sum of the direct wave and the scattered field from the distribution of scatterers

$$\Psi(\mathbf{r})=\Psi_0(\mathbf{r})+\sum_{i=1}^{n} G^{(0)}(\mathbf{r},\mathbf{r}_i)A_i\Psi(\mathbf{r}_i). \tag{A1}$$

Replacing the incident wave field at scatterer i by the homogeneous incident wave field corresponds to the Born approximation (BORN):

$$\Psi(\mathbf{r})=\Psi_0(\mathbf{r})+\sum_{i=1}^{n} G^{(0)}(\mathbf{r},\mathbf{r}_i)A_i\Psi_0(\mathbf{r}_i). \tag{A2}$$

However, the total wave field at each scatterers consist of the incident wave field plus the scattered wave field radiated from all other scatterers:

$$\Psi(\mathbf{r}_i)=\Psi_0(\mathbf{r}_i)+\sum_{\substack{j=1\\ j\neq i}}^{n} G^{(0)}(\mathbf{r}_i,\mathbf{r}_j)A_j\Psi(\mathbf{r}_j). \tag{A3}$$

Define the vector Ψ by

$$\boldsymbol{\Psi}=\begin{pmatrix}\Psi(\mathbf{r}_1)\\ \Psi(\mathbf{r}_2)\\ \cdots\\ \cdots\\ \Psi(\mathbf{r}_n)\end{pmatrix} \tag{A4}$$

and let the matrix $\mathbf{M}$ with rank number equal to the number of scatterers be given by

$$\mathbf{M}=\begin{pmatrix} -1 & A_2G^{(0)}(\mathbf{r}_1,\mathbf{r}_2) & \cdots & \cdots & A_nG^{(0)}(\mathbf{r}_1,\mathbf{r}_n)\\ A_1G^{(0)}(\mathbf{r}_2,\mathbf{r}_1) & -1 & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ A_1G^{(0)}(\mathbf{r}_n,\mathbf{r}_1) & \cdots & \cdots & \cdots & -1 \end{pmatrix}. \tag{A5}$$

With a corresponding definition of a vector Ψ_0 in Eq. (A4), Eq. (A3) can then compactly be written as

$$\mathbf{M}\boldsymbol{\Psi}=\boldsymbol{\Psi}_0 \quad \text{or} \quad \boldsymbol{\Psi}=-\mathbf{M}^{-1}\boldsymbol{\Psi}_0. \tag{A6}$$

Numerical methods can be used to calculate $\boldsymbol{\Psi}$ and consequently $\Psi(\mathbf{r})$ from Eq. (A1). This solution contains all multiple isotropic point scatterer interactions and will be denoted as the multiple isotropic point scattering solution (MIPS). In case of the Born approximation, Eq. (A2) is used to calculate the wave field. Note that the expressions derived in this appendix are equally valid in any dimension and for any incident wave $\Psi_0(\mathbf{r})$.

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